MATH 121 **Final Exam Solution (version II)** **Radford** **12/03/02**

1. Let $y = f(x) = 6 + \sqrt{5x-7}$. The calculation

$$\begin{aligned} y &= 6 + \sqrt{5x-7}, \\ y-6 &= \sqrt{5x-7}, \\ (y-6)^2 &= 5x-7, \\ (y-6)^2+7 &= 5x, \\ \frac{(y-6)^2+7}{5} &= x \ \ (\textbf{15 points}) \end{aligned}$$

shows that the inverse of $f(x)$ is given by $f^{-1}(x) = \dfrac{(x-6)^2+7}{5}$ (**5 points**).

2. Since $f(x) = x^3 + 2x - 3$ has integer coefficients, the rational roots of $f(x)$ are among $\pm 1, \pm 3$. Since $f(1) = 0$ it follows that $x = 1$ is a root of $f(x)$. Therefore $x - 1$ divides $f(x)$. Dividing $x - 1$ into $f(x)$ gives the quotient of $x^2 + x + 3$ and remainder 0. Thus

$$x^3 + 2x - 3 = (x-1)(x^2 + x + 3)$$

By the quadratic formula the roots of $x^2 + x + 3$ are $\dfrac{-1 \pm \sqrt{11}\imath}{2}$. Thus the roots of $f(x)$ are

$$1 \ \ (\textbf{4 points}), \quad \frac{-1+\sqrt{11}\imath}{2} \ \ (\textbf{4 points}), \quad \frac{-1-\sqrt{11}\imath}{2} \ \ (\textbf{4 points}).$$

As a product of linear factors,

$$f(x) = (x-1)\left(x - \frac{-1+\sqrt{11}\imath}{2}\right)\left(x - \frac{-1-\sqrt{11}\imath}{2}\right) \ \ (\textbf{8 points}).$$

3. The height function is given by $h(t) = -16t^2 + 10t + 500$ feet.

(a) Rewriting $h(t)$ as

$$-16t^2 + 10t + 500 = -16(t^2 - \frac{10}{16}t) + 500 = -16(t - \frac{10}{32})^2 + (500 + 16\left(\frac{10}{32}\right)^2)$$

we see that the maximal altitude is ≈ 501.563 *feet* (**10 points**).

(b) We want to solve $h(t) = 0$, where $t \geq 0$, or $-16t^2 + 10t + 500 = 0$, where $t \geq 0$. By the quadratic formula $t = \dfrac{-10 \pm \sqrt{10^2 - (4)(-16)(500)}}{2(-16)}$, so $t \approx 5.911$ *seconds.* (**10 points**).

4. (a) The calculation

$$2\sin^2(x) + \cos(x) = 2,$$

$$
\begin{aligned}
2(1-\cos^2(x))+\cos(x) &= 2,\\
-2\cos^2(x)+\cos(x) &= 0,\\
\cos(x)\,(-2\cos(x)+1) &= 0,
\end{aligned}
$$

shows that the set of solutions to the original equation is the set of solutions to $\cos(x)=0$ together with the set of solutions to $\cos(x)=\frac{1}{2}$ (**3 points**). Thus *all* solutions the original equation are described by

$$x=\frac{\pi}{2}+\pi n \ \ (\textbf{3 points}),\quad \frac{\pi}{3}+2\pi n \ \ (\textbf{3 points}),\quad \text{or}\ \ \frac{5\pi}{3}+2\pi n \ \ (\textbf{3 points}),$$

where n is any integer.

(b) The calculation

$$
\begin{aligned}
\log\sqrt{x^2-10} &= 3,\\
\sqrt{x^2-10} &= 10^3,\\
x^2-10 &= (10^3)^2,\\
x^2 &= 10^6+10,\\
x &= \pm\sqrt{10^6+10},
\end{aligned}
$$

the steps of which are reversible, shows that $x=\sqrt{10^6+10}$ (**4 points**), $-\sqrt{10^6+10}$ (**4 points**).

5. (a) $600\left(1+\dfrac{0.08}{4}\right)^{4\cdot 9}$ (**5 points**) ≈ 1044.615 dollars (**5 points**).

(b) Let P be the initial amount invested at $100r\%$ per year compounded continuously. Then $2P=P(15)=Pe^{r15}$ which means $2=e^{r15}$ (**3 points**) since $P>0$. Therefore $r=\dfrac{\ln 2}{15}$ (**3 points**) $\approx .046$ (**4 points**) which can be expressed as 4.6%.

6. a) All real numbers except for $x=9/17$ (**3 points**). b) $y=5/17$ (**3 points**). c) $x=-9/17$ (**3 points**). d) At $x=-3/5$ (**3 points**). e) At $y=-1/3$ (**3 points**). f) Labeling (**3 points**) and shape of graph (**2 points**).

7. (a) *no* (**1 points**). Take $x=\frac{\pi}{4}$ for example. In this case $\cot(x+\pi)=\cot(\frac{\pi}{4}+\pi)=1$ and $-\cot(x)=-\cot(\frac{\pi}{4})=-1$ (**3 points**).

(b) *no* (**1 points**). Take $x=0$ for example. In this case $e^{x+5}=e^5$ and $e^x+e^5=e^0+e^5=1+e^5$ (**3 points**).

(c) *yes* (**4 points**).

(d) *no* (**1 points**). Take $x=1$ for example. In this case $\ln(x+10)=\ln(11)\neq 0$ and $(\ln 1)(\ln 10))=(0)\ln(10)=0$ (**3 points**).

(e) *no* (**1 points**). Take $x=2\pi$ for example. In this case $\arccos(\cos(2\pi))=\arccos(\cos(0))=0$, since $0\leq x\leq\pi$, which is not $x=2\pi$ (**3 points**).

8. Since $\tan(t) = -\dfrac{7}{6}$ and $\pi < t < 2\pi$, it follows that the terminal side of t is in the fourth quadrant. Therefore $(6, -7)$ is a point on the terminal side of the angle t. Since $r = \sqrt{6^2 + (-7)^2} = \sqrt{85}$ we conclude that

$$\cos(t) = \frac{6}{\sqrt{85}}, \qquad \sin(t) = -\frac{7}{\sqrt{85}}, \qquad \tan(t) = \frac{\sin(t)}{\cos(t)} = -\frac{7}{6}.$$

The last calculation is not necessary, of course, but does provide a consistency check. Thus

$$(a)\cos(t) = \frac{6}{\sqrt{85}}\ (\mathbf{5\ points}), \quad (b)\csc(t) = -\frac{\sqrt{85}}{7}\ (\mathbf{5\ points}),$$

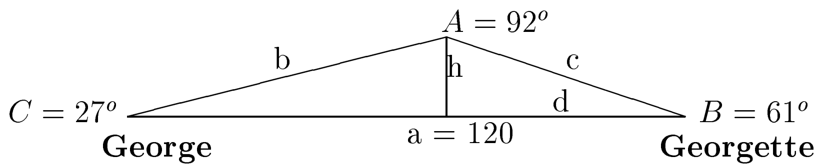
and since $\dfrac{\pi}{2} < \dfrac{t}{2} < \pi$,

$$(c)\sin(t/2) = \sqrt{\frac{1-\cos(t)}{2}} = \sqrt{\frac{1-\frac{6}{\sqrt{85}}}{2}} = \sqrt{\frac{\sqrt{85}-6}{2\sqrt{85}}}\ (\mathbf{5\ points}),$$

and finally

$$(d)\cos(2t) = \cos^2(t) - \sin^2(t) = \left(\frac{6}{\sqrt{85}}\right)^2 - \left(\frac{-7}{\sqrt{85}}\right)^2 = \frac{36}{85} - \frac{49}{85} = -\frac{13}{85}\ (\mathbf{5\ points}).$$

9. (a) Let A be the angle formed by the line segments from George to the top of the streetlight to Georgette, let $B = 61^o$, and let $C = 27^o$. Then $A = 180^o - 27^o - 61^o = 92^o$. Below is a picture *which is not meant to be to scale.*



By the Law of Sines

$$\frac{120}{\sin 92^o} = \frac{b}{\sin 61^o}.$$

Therefore

$$b = \left(\frac{120}{\sin 92^o}\right)\sin 61^o \approx 105.018\ feet\ \ (\mathbf{4\ points}).$$

Consequently the height of the streetlight is $h = b\sin 27^o \approx 47.677\ feet$ (**8 points**).

(b) The distance from Georgette to the streetlight is

$$d = h \cot 61^o = \frac{h}{\tan 61^o} \approx 26.428 \; feet \quad (\mathbf{8\ points}).$$

10. (a) The modulus of $1 + \sqrt{3}\imath$ is $|1 + \sqrt{3}\imath| = \sqrt{1^2 + (\sqrt{3})^2} = 2$. Therefore $1 + \sqrt{3}\imath = 2\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\imath\right) = 2\left(\cos(\frac{\pi}{3}) + \imath \sin(\frac{\pi}{3})\right)$ (**10 points**).

(b) Using the polar form of $1 + \imath$ we calculate

$$\begin{aligned}
(1+\sqrt{3}\imath)^7 &= \left(2\left(\cos(\frac{\pi}{3}) + \imath \sin(\frac{\pi}{3})\right)\right)^7 \\
&= 2^7 \left(\cos(\frac{\pi}{3}) + \imath \sin(\frac{\pi}{3})\right)^7 \\
&= 2^7 \left(\cos(\frac{7\pi}{3}) + \imath \sin(\frac{7\pi}{3})\right) \\
&= 128 \left(\cos(\frac{\pi}{3}) + \imath \sin(\frac{\pi}{3})\right) \quad (\mathbf{10\ points}).
\end{aligned}$$

Note that $(1 + \sqrt{3}\imath)^7 = 64(1 + \sqrt{3}\imath)$.