Math 215: **Hour Exam 2** (Solutions)
Prof. S. Smith Friday 5 April 2002

Each problem is worth 25 points. (Some problems with parts indicate subdivision of points).
Notation: $\mathbf{N}$ denotes the natural numbers $1, 2, 3, \ldots$; $\mathbf{Z}$ is the integers; $\mathbf{R}$ is the real numbers.

Problem 1:
(a) (10 points) As usual π means $3.1415\ldots$. Let P denote the statement:

$$(\forall x \in \mathbf{Z})\ (\exists y \in \mathbf{Z})\ ,\ x + y = \pi .$$

(i) Use the rules to write out not-P (so quantifiers come first, with "not" later).
(i) not-P becomes : $(\exists x \in \mathbf{Z})\ (\forall y \in \mathbf{Z})\ ,\ x + y \neq \pi$.
(ii) Which is true, P or not-P ? (Why?)
(ii) not-P is true; PROOF: Since $x, y \in \mathbf{Z}$, also $x + y \in \mathbf{Z}$; so $x + y \neq \pi$ as $\pi \notin \mathbf{Z}$.
(b) (15 points: New problem, forget about P above) Prove the new statement:

$$(\forall x \in \mathbf{Z})\ ,\ [\ (\exists y \in \mathbf{Z}, x = 2y)\ \Rightarrow \text{not-}(\ (\exists z \in \mathbf{Z}), x^2 = 2z + 1\ )\ ]\ .$$

What does this statement say in the usual language of even and odd integers ?
Probably easiest by contradiction:
Given: $[(\exists y), x = 2y]$ and $[(\exists z), x^2 = 2z + 1]$; Goal: contradiction.
PROOF: $4y^2 = x^2 = 2z + 1$, so $1 = 4y^2 - 2z = 2(2y^2 - z)$ is even, contradicting that 1 is odd.
In other words, again: "The square of an even integer is even."

Problem 2:
Define $f : \mathbf{R} \to \mathbf{R}$ by the formula $f(x) = 5x - 7$.
(a) (10 points) Is f injective ? (Why/why not)
Yes. PROOF: If $5x - 7 = 5y - 7$, then $5x = 5y$, and so $x = y$.
(b) (10 points) Is f surjective ? (Why/why not)
Yes. PROOF: Solve $y = 5x - 7$ for x; that is, $5x = y + 7$ so $x = \frac{y+7}{5}$.
(c) (5 points) Is f invertible ? (If not, why not; if so, give f^{-1})
Yes. PROOF: By (a) and (b), f is bijective. By (b), $f^{-1}(x) = \frac{x+7}{5}$.

Problem 3: For sets A, B, and C:
(a) (10 points) Assume that $|A| = 3$ and $|B| = 4$.
What is the number $|Fun(A, B)|$ of functions from A to B ?
A function $f : A \to B$ can be given as a list of values $(f(a_1), f(a_2), f(a_3))$ where each $f(a_i) \in B$; so the number of choices is $4^3 = 64$.
(b) (15 points) Assume that:

$$|A| = 5,\ |B| = 6,\ |C| = 7;\ |A \cap B| = 4,\ |A \cap C| = 3,\ |B \cap C| = 2;\ \text{and}\ |A \cap B \cap C| = 1.$$

What is $|A \cup B \cup C|$? (Why?)
By inclusion/exclusion, $|A \cup B \cup C| = (5 + 6 + 7) - (4 + 3 + 2) + 1 = 18 - 9 + 1 = 10$.
My numbers were inconsistent: reverse 4,3,2 to 2,3,4 in order to make the Venn diagram all nonnegative.

Problem 4:

Let M denote the set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with coefficients $a, b, c, d \in \mathbf{Z}$.

Show that M is a denumerable set (that is, countably infinite).

(Hint: Give a bijection of M with $\mathbf{Z}^m$ for a suitable $m > 1$. Then apply known theorems.)

PROOF: The map taking $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *to* (a, b, c, d) *is a bijection (easy to prove) from* M *to* $\mathbf{Z}^4$.

By standard theorems: $\mathbf{Z}$ *is denumerable, so the Cartesian product* $\mathbf{Z}^4$ *is denumerable;*
so M *is denumerable.*