Math 215: **Final Exam**

Prof. S. Smith 1–3 pm, LC D1  
Tuesday 30 April 2002

Each of the 4 problems is worth 50 points. (Some with parts indicate subdivision of points).

**Notation:** $\mathbb{N}$ denotes the natural numbers $1, 2, 3, \ldots$; $\mathbb{Z}$ is the integers; $\mathbb{R}$ is the real numbers.

**Problem 1:** Define a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x_1, x_2) = (x_2, x_1 + x_2)$.

(a) (25 points) Is $f$ injective? You must PROVE your answer.

Yes: For general $(a_1, a_2)$ and $(b_1, b_2)$, we assume $f(a_1, a_2) = f(b_1, b_2)$, and we must show this forces $(a_1, a_2) = (b_1, b_2)$.

**PROOF:** We are assuming $(a_2, a_1 + a_2) = (b_2, b_1 + b_2)$; so $a_2 = b_2$ and $a_1 + a_2 = b_1 + b_2$.

Substituting the first equation into the second, we get $a_1 = b_1$.

So our assumption implies $(a_1, a_2) = (b_1, b_2)$, and hence $f$ is injective.

(b) (25 points) Is $f$ surjective? You must PROVE your answer.

Yes: For general $(y_1, y_2)$, we assume there is some $(x_1, x_2)$ with $f(x_1, x_2) = (y_1, y_2)$, and we must solve for $(x_1, x_2)$ in terms of $(y_1, y_2)$.

**PROOF:** We are assuming $(x_2, x_1 + x_2) = (y_1, y_2)$, so $x_2 = y_1$ and $x_1 + x_2 = y_2$.

Substituting the first equation into the second, we get $x_1 = -y_1 + y_2$.

So we do get a solution $(x_1, x_2) = (-y_1 + y_2, y_1)$, and hence $f$ is surjective.

**Problem 2:** Let $S$ denote the set of $3 \times 3$ “skew-symmetric” matrices over the integers $\mathbb{Z}$:

These are the matrices of form $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ where $a, b, c \in \mathbb{Z}$.

Show that $S$ is denumerable (i.e., countably infinite). You may quote standard theorems.

Define a map $f : S \to \mathbb{Z}^3$ taking the above matrix to $(a, b, c)$.

Then $f$ is easily checked to be a bijection. So by standard theorems:

First $\mathbb{Z}$ is denumerable, then the Cartesian product $\mathbb{Z}^3$ is denumerable, and so we conclude $S$ is denumerable.
Problem 3: Three customers each purchase one television from 4 possible models.
(a) (25 points) How many possible purchases are there?
(i.e., keeping track of which customer purchased what)
\[
\text{The set } C \text{ of customers has size } |C| = 3, \\text{ and the set } T \text{ of choices for a model of a television has size } |T| = 4. 
\]
\[
The \text{ purchases are functions } \text{Fun}(C,T), \text{ or more simply just lists, i.e. triples in } T^3. 
\]
\[
\text{Thus the number of purchases is } |T|^{|C|} \text{ or } |T|^3, \text{ namely } 4^3 = 64. 
\]
(b) (25 points) How many possible purchases are there, if we only keep track of how many of each model was purchased? (that is, NOT keeping track of which customer ordered which model)
Show work.

There are 3 cases: all 3 customers order the same model; or 2 customers order one model while the third customer orders a different model; or all 3 customers order different models:

\[\text{Same: There are } \binom{4}{1} = 4 \text{ choices of one model.}\]
\[\text{2 and 1: There are } \binom{4}{1} = 4 \text{ choices of a model ordered by 2 customers;}\]
\[\text{then } \binom{3}{1} = 3 \text{ choices for the model ordered by the remaining customer,}\]
\[\text{hence } 4 \cdot 3 = 12 \text{ choices in this case.}\]

\[\text{All different: There are } \binom{4}{3} = 4 \text{ choices of 3 different models.}\]
\[\text{Hence there are } 4 + 12 + 4 = 20 \text{ different possible purchases.}\]

Problem 4:
(a) (25 points) Show that if a positive integer \( n \) is a perfect square (that is, \( n = a^2 \) for \( a \in \mathbb{Z} \)), then \( n \) has remainder 0, 1, or 4 when divided by 5. Conclude 111111113 is not a perfect square.

\[\text{Since } a \text{ has remainder 0,1,2,3, or 4, and these numbers square to 0,1,4,9,16,}\]
\[\text{we see that } n = a^2 \text{ has remainder 0,1,4,1; that is, 0,1, or 4.}\]

\[\text{So since 111111113 has remainder 3, it is not a perfect square.}\]

(b) (25 points) Use the Euclidean algorithm to find the greatest common divisor \( d = gcd(210, 112) \).
Then for that number \( d \), find \( m, n \in \mathbb{Z} \) such that \( d = 210m + 112n \).

\[\text{Euclidean algorithm:}\]
\[210 = 112 \cdot 1 + 98\]
\[112 = 98 \cdot 1 + 14\]
\[98 = 14 \cdot 7 + 0, \text{ so that } 14 = gcd(210, 112).\]
Then by the “starting at the bottom” method:
\[14 = 112 - 98 = 112 \cdot 1 - (210 \cdot 1 - 112 \cdot 1) = 210 \cdot (-1) + 112 \cdot 2\]
so one possibility is \( m = -1, n = 2.\)