Math 215: **Final Exam** (Solutions)
Prof. S. Smith 1–3 pm, LC D1 Tuesday 30 April 2002

Each of the 4 problems is worth 50 points. (Some with parts indicate subdivision of points).
Notation: $\mathbf{N}$ denotes the natural numbers $1, 2, 3, \ldots$; $\mathbf{Z}$ is the integers; $\mathbf{R}$ is the real numbers.

Problem 1: Define a function $f : \mathbf{R}^2 \to \mathbf{R}^2$ by $f(x_1, x_2) = (x_2, x_1 + x_2)$.

(a) (25 points) Is f injective ? You must PROVE your answer.

Yes: For general (a_1, a_2) and (b_1, b_2),
we assume $f(a_1, a_2) = f(b_1, b_2)$, and we must show this forces $(a_1, a_2) = (b_1, b_2)$.
PROOF: We are assuming $(a_2, a_1 + a_2) = (b_2, b_1 + b_2)$; so $a_2 = b_2$ and $a_1 + a_2 = b_1 + b_2$.
Substituting the first equation into the second, we get $a_1 = b_1$.
So our assumption implies $(a_1, a_2) = (b_1, b_2)$, and hence f is injective.

(b) (25 points) Is f surjective ? You must PROVE your answer.

Yes: For general (y_1, y_2), we assume there is some (x_1, x_2) with $f(x_1, x_2) = (y_1, y_2)$,
and we must solve for (x_1, x_2) in terms of (y_1, y_2).
PROOF: We are assuming $(x_2, x_1 + x_2) = (y_1, y_2)$, so $x_2 = y_1$ and $x_1 + x_2 = y_2$.
Substituting the first equation into the second, we get $x_1 = -y_1 + y_2$.
So we do get a solution $(x_1, x_2) = (-y_1 + y_2, y_1)$, and hence f is surjective.

Problem 2: Let S denote the set of 3×3 "skew-symmetric" matrices over the integers $\mathbf{Z}$:

These are the matrices of form $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$ where $a, b, c \in \mathbf{Z}$.

Show that S is denumerable (i.e., countably infinite). You may quote standard theorems.

Define a map $f : S \to \mathbf{Z}^3$ taking the above matrix to (a, b, c).
Then f is easily checked to be a bijection. So by standard theorems:
First $\mathbf{Z}$ is denumberable, then the Cartesian product $\mathbf{Z}^3$ is denumerable,
and so we conclude S is denumerable.

Problem 3: Three customers each purchase one television from 4 possible models.
(a) (25 points) How many possible purchases are there ?
(i.e., keeping track of which customer purchased what)

The set C of customers has size $|C| = 3$,
and the set T of choices for a model of a television has size $|T| = 4$.
The purchases are functions $Fun(C, T)$, or more simply just lists, i.e. triples in T^3.
Thus the number of purchases is $|T|^{|C|}$ or $|T|^3$, namely $4^3 = 64$.

(b) (25 points) How many possible purchases are there, if we only keep track of how many of each model was purchased ? (that is, NOT keeping track of which customer ordered which model) Show work.

There are 3 cases: all 3 customers order the same model; or 2 customers order one model while the third customer orders a different model; or all 3 customers order different models:
Same: There are $\binom{4}{1} = 4$ choices of one model.
2 and 1: There are $\binom{4}{1} = 4$ choices of a model ordered by 2 customers;
then $\binom{3}{1} = 3$ choices for the model ordered by the remaining customer,
hence $4 \cdot 3 = 12$ choices in this case.
All different: There are $\binom{4}{3} = 4$ choices of 3 different models.
Hence there are $4 + 12 + 4 = 20$ different possible purchases.

Problem 4:
(a) (25 points) Show that if a positive integer n is a perfect square (that is, $n = a^2$ for $a \in \mathbf{Z}$), then n has remainder 0, 1, or 4 when divided by 5. Conclude 111111113 is not a perfect square.

Since a has remainder 0,1,2,3, or 4, and these numbers square to 0,1,4,9,16,
we see that $n = a^2$ has remainder 0,1,4,4,1; that is, 0,1, or 4.
So since 111111113 has remainder 3, it is not a perfrect square.

(b) (25 points) Use the Euclidean algorithm to find the greatest common divisor $d = gcd(210, 112)$. Then for that number d, find $m, n \in \mathbf{Z}$ such that $d = 210m + 112n$.

Euclidean algorithm:
$210 = 112 \cdot 1 + 98$
$112 = 98 \cdot 1 + 14$
$98 = 14 \cdot 7 + 0$, *so that* $14 = gcd(210, 112)$.
Then by the "starting at the bottom" method:
$14 = 112 - 98 = 112 \cdot 1 - (210 \cdot 1 - 112 \cdot 1) = 210 \cdot (-1) + 112 \cdot 2$
so one possibility is $m = -1$, $n = 2$.