An Invitation to Higher Mathematics
Math 215, Fall Semester, 2001

Midterm Exam – October 5 – Solutions

1. (25 points) Prove by contradiction that there do not exist integers m, n such that
$$9m + 15n = 25$$

Proof: Assume there exist integers m, n such that $9m + 15n = 25$ and we show this leads to a contradiction. By assumption, there are integers m, n so that

$$25 = 9m + 15n = 3 \cdot 3m + 3 \cdot 5n = 3 \cdot (3n + 5m) = 3p$$

where p is an integer. But also, $25 = 3 \cdot 8 + 1$ so $3 \cdot 8 + 1 = 3p$. This implies $1 = 3(p - 8)$ which is impossible for p an integer. This contradiction implies that no such integer solutions m, n can exist. □

2. (25 points) Use the method of induction to prove that for all $n \geq 1$,
$$\sum_{i=0}^{n} 2^i \; = \; 2^{n+1} - 1$$

Proof: We formulate the inductive statement first.

P(n): $\displaystyle\sum_{i=0}^{n} 2^n \; = \; 2^{n+1} - 1$

We prove first the case for $n = 1$, $2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$.

Next, we assume that P(n) is true, and use this to prove the statement P(n+1). We calculate the LHS of P(n+1) using the inductive hypotheses

$$\begin{aligned} 2^0 + 2^1 + \cdots + 2^n + 2^{n+1} &= (2^0 + 2^1 + \cdots + 2^n) + 2^{n+1} \\ &= (2^{n+1} - 1) + 2^{n+1} \\ &= 2 \cdot 2^{n+1} - 1 \\ &= 2^{n+2} - 1 \end{aligned}$$

which is the conclusion of P(n+1).

By the Principle of Induction, it follows that P(n) is true for all integers $n \geq 1$. □

3. (25 points) Use the method of cases to prove that for all sets A, B, C

$$A \times (B \cup C) \;=\; (A \times B) \cup (A \times C)$$

Proof: We first show $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.
Suppose $(x, y) \in A \times (B \cup C)$, then $x \in A$ and $y \in B \cup C$.
Case 1) If $y \in B$, then $(x, y) \in A \times B$ so $(x, y) \in (A \times B) \cup (A \times C)$.
Case 2) If $y \in C$, then $(x, y) \in A \times C$ so $(x, y) \in (A \times B) \cup (A \times C)$.
In both cases, $(x, y) \in (A \times B) \cup (A \times C)$, so
$A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.

We next show the other inclusion $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$.
Suppose $(x, y) \in (A \times B) \cup (A \times C)$.
Case 1) If $(x, y) \in A \times B$, then $x \in A$ and $y \in B$ so $y \in B \cup C$ and $(x, y) \in A \times (B \cup C)$.
Case 2) If $(x, y) \in A \times C$, then $x \in A$ and $y \in C$ so $y \in B \cup C$ and $(x, y) \in A \times (B \cup C)$.
In both cases, $(x, y) \in A \times (B \cup C)$, so $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$.
These two inclusions combine to prove the two sets are equal. □

4. (25 points) For a function $f : A \to B$

a) Give a precise definition using quantifiers of "f is injective".

Solution:

$$\forall x, y \in A, x \neq y \implies f(x) \neq f(y)$$

b) Give a precise definition using quantifiers of "f is surjective".

Solution:

$$\forall y \in B, \exists x \in A, y = f(x)$$

Now suppose $f : \mathbb{R} \to \mathbb{R}$ is the function defined by $f(x) = x^3 - x$.

c) Determine whether or not f is injective.

Solution:

$$f(x) = 0 \iff x^3 - x = 0 \iff x(x+1)(x-1) = 0 \iff x \in \{-1, 0, 1\}$$

so $f(x)$ is not injective, as $f(-1) = f(0) = f(1) = 0$.

d) Does f have an inverse function? If so, give a formula for $f^{-1}(y)$.

Solution: f is not injective, so cannot be invertible.