An Invitation to Higher Mathematics
Math 215, Fall Semester, 2001
Instructor: Steve Hurder

Hour Exam 2 – November 12 - Solutions

1. Use Pascal's Triangle to find the coefficient of x^3y^5 in the expansion of $(3x - y)^8$.

Solution: From Pascal's Triangle, "8 choose 3" is 56, so the term containing x^3y^5 is

$$56 \cdot (3x)^3 \cdot (-y)^5 = 56 \cdot 27 \cdot (-1) \cdot x^3y^5 = -1512x^3y^5$$

The answer is -1512. $\square$

2. A and B are sets.

a) Give the definitions of the sets $A \times B$ and $B \times A$.

Solution:

$$A \times B = \{(x, y) \mid x \in A, y \in B\}$$

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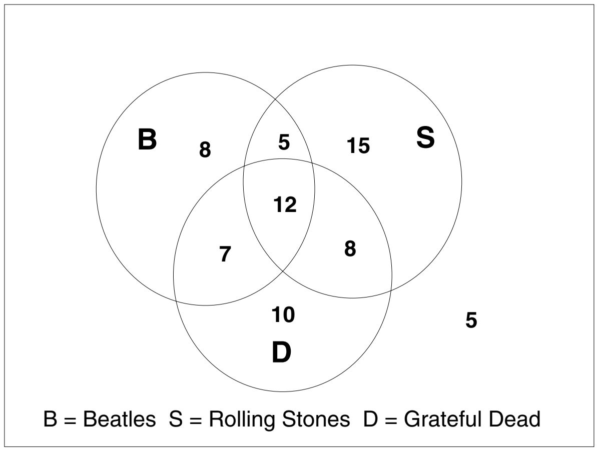
b) Prove there is a bijection between $A \times B$ and $B \times A$. (Give a map between $A \times B$ and $B \times A$, and show this map is a bijection.)

Solution: Define $f : A \times B \to B \times A$ by $f(x, y) = (y, x)$.
We show f is a bijection.
f is surjective: given any $(b, a) \in B \times A$, then $(a, b) \in A \times B$ and $f(a, b) = (b, a)$.
f is injective: if $f(x, y) = f(x', y')$ then $(y, x) = f(x, y) = f(x', y') = (y', x')$. A pair is equal if and only if both entries are equal, so $y = y'$ and $x = x'$. Then $(x, y) = (x', y')$. $\square$

3. At a recent high school 30^{th} reunion, 70 former classmates were recalling three great rock concerts which some of them had attended. The concerts were by the Beatles (B), the Grateful Dead (D) and the Rolling Stones (S). 32 people recalled attending to the Beatles concert, 37 attended the Dead concert, and 40 attended the Stones concert. 12 people attended all 3 concerts, while 17 attended both the Beatles and the Stones concerts, 19 attended both the Beatles and the Dead concerts, and 20 attended both the Dead and the Stones concerts.

a) Draw a Venn diagram representing the people who attended the various concerts.



b) How many people attended just 2 of the 3 concerts?

Solution: 20.

c) How many people attended none of the concerts?

Solution: 5. $\square$

4. A, B and C are sets. Suppose that $f\colon A \to B$ is a surjective function, and the function $g\colon B \to C$ is *not* injective. Prove that $g \circ f\colon A \to C$ is *not* injective.

Solution: We will show there exists $a \neq a' \in A$ with $g(f(a)) = g(f(a'))$. This proves $g \circ f$ is not injective.

g is not injective, so there exists $b \neq b' \in B$ with $g(b) = g(b')$.

f is surjective, so there exists $a \in A$ with $f(a) = b$, and also $a' \in A$ with $f(a') = b'$.

The choice of a, a' gives $g(f(a)) = g(b) = g(b') = g(f(a'))$.

If $a = a'$ then $f(a) = f(a') \implies b = b'$ contradicting the choice of $b \neq b'$. So $a \neq a'$. This shows $g \circ f$ is not injective. $\square$

5. Three people, named X, Y and Z, do an experiment. Each person has a basket containing 6 types of fruits – apples, bananas, cherries, grapes, oranges, and pomegranates. To do the experiment, each person is asked to choose a fruit from his or her basket. The result is a choice by each of X, Y and Z of one of the fruits.

   a) How many possible results are there for the experiment?

Solution: Each person has 6 choices of fruit. The choices are made from different baskets, so their choices of fruit are independent. The number of ways they can choose is then $6 \times 6 \times 6 = 216$.

Another way to explain this, is that when the three people make a choice, we get a function from the set of people $A = \{X, Y, Z\}$ to the set of fruit F. The set A has 3 elements, the set B has 6, so the number of possible functions from A to B is $3^6 = 216$.

   b) How many times must the experiment be repeated to be sure the same result is obtained twice?

Solution: 217. There are 216 possible outcomes, corresponding to the possible functions from A to B. By the Pigeonhole Principle, if we repeat the experiment 217 times, at least 2 of the experiments must give the same result. $\square$