An Invitation to Higher Mathematics
Math 215, Fall Semester, 2001

Solutions to Problems & Exercises
Week 2 – August 27-31

7. Prove by contradiction that there do not exist integers m and n such that

$$14m + 21n = 100$$

Proof: We give a proof by contradiction. That is, we assume there exists integers m and n so that $14m + 21n = 100$ and derive from this a contradiction. It will follow that there cannot exist integers n and m satisfying the equation.

We notice that both 14 and 21 are divisible by 7, while 100 is not. We use this to obtain a contradiction. Rewrite the LHS of the equation as

$$14m + 21n = 7 \cdot 2m + 7 \cdot 3n = 7 \cdot (2m + 3n)$$

So $100 = 14m + 21n = 7 \cdot (2m + 3n)$ where $2m + 3n$ is an integer. This shows that 100 is divisible by 7, which is a contradiction. □

8. Prove by contradiction that for any integer n,

$$n^2 \text{ is odd } \implies n \text{ is odd}$$

Proof: We will assume that n is an integer such that n^2 is odd and n is even, and derive from this a contradiction. Hence, no such integer n can exist, so for every n we have n^2 odd implies that n is odd.
The assumption n is even implies $n = 2p$ for some integer p.
Then $n^2 = (2p)^2 = 2 \cdot 2p^2$ where $2p^2$ is an integer, so n^2 is even.
This contradicts the assumption n^2 is odd. □

9. (*turn in Wednesday, September 5*)
Prove by contradiction that for any integer n,

$$n^2 \text{ is even } \implies n \text{ is even}$$

Hint: An integer n is odd if and only if $n = 2q + 1$ for some integer q.

Proof: We will assume that n is an integer such that n^2 is even and n is odd, and derive from this a contradiction. Hence, no such integer n can exist, so for every n we have n^2 even implies that n is even.
By the hint, the assumption n is odd implies $n = 2q + 1$ for some integer q.
Then $n^2 = (2q + 1)^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1$ where $2q^2 + 2q$ is an integer.
This shows n^2 is odd, contradicting the assumption n^2 is even. □

10. (*turn in Wednesday, September 5*)
Prove that, for all real numbers a and b,

$$|a+b| \leq |a| + |b|$$

Proof: We consider four cases, corresponding to a and b non-negative or negative.
Case 1. $a \geq 0$ and $b \geq 0$. Then $a+b \geq 0$ so

$$|a+b| = a+b = |a| + |b|$$

Case 2. $a \geq 0$ and $b < 0$. This case has two subcases, $a+b \geq 0$ and $a+b < 0$.
If $a+b \geq 0$ then using that $b < 0$ so $b < -b$ and $-b = |b|$ we have

$$|a+b| = a+b < a-b = |a| + |b|$$

If $a+b < 0$ then using that $-a \leq 0$ so $-a \leq a$ and $-b = |b|$ we have

$$|a+b| = -(a+b) = -a + -b \leq a - b = |a| + |b|$$

Case 3. $a < 0$ and $b \geq 0$. This case has two subcases, $a+b \geq 0$ and $a+b < 0$.
If $a+b \geq 0$ then using that $a < 0$ so $a < -a$ and $-a = |a|$ we have

$$|a+b| = a+b < -a+b = |a| + |b|$$

If $a+b < 0$ then using that $-b \leq 0$ so $-b \leq b$ and $-a = |a|$ we have

$$|a+b| = -(a+b) = -a + -b \leq -a + b = |a| + |b|$$

Case 4. $a < 0$ and $b < 0$. Then $a+b < 0$ so

$$|a+b| = -(a+b) = -a - b = |a| + |b|$$

$\square$

11. Prove the following statements concerning positive integers a, b, and c.
(i) (a divides b) and (a divides c) $\Longrightarrow$ a divides $(b + c)$
Proof:
a divides b implies $b = a \cdot p$ for some integer p.
a divides c implies $c = a \cdot q$ for some integer q.
This implies $b + c = a \cdot p + a \cdot q = a(p + q)$ where $p + q$ is an integer.
Thus, a divides $b + c$. □

(ii) (a divides b) or (a divides c) $\Longrightarrow$ a divides (bc)
Proof: There are two cases, either a divides b or a does not divide b.
In the second case, by assumption we must have a divides c.
Case 1. a divides b implies $b = a \cdot p$ for some integer p.
This implies $b \cdot c = (a \cdot p) \cdot c = a \cdot (p \cdot c)$ where $p \cdot c$ is an integer. Thus, a divides bc.
Case 2. a divides c implies $c = a \cdot q$ for some integer q.
This implies $b \cdot c = b \cdot (a \cdot q) = a \cdot (b \cdot q)$ where $b \cdot q$ is an integer. Thus, a divides bc. □

12. Which of the following conditions are necessary for the positive integer n to be divisible by 6?
Solution: For all of the following questions, we use that 6 divides n implies
$n = 6 \cdot p = 2 \cdot 3 \cdot p$ for some integer p.

(i) 3 divides n. *Yes:* $n = 3 \cdot 2p$ so must be divisible by 3.

(ii) 9 divides n. *No:* $2p$ need not be divisible by 3. For example, take $n = 6$.

(iii) 12 divides n. *No:* p need not be divisible by 2. For example, take $n = 6$.

(iv) $n = 12$. *No:* $n = 6$ is divisible by 6.

(v) 6 divides n^2. *Yes:* $n^2 = n \cdot n \; 6 \cdot p \cdot n = 6 \cdot pn$.

(vi) 2 divides n and 3 divides n. *Yes:* $n = 2 \cdot 3p = 3 \cdot 2p$.

(vii) 2 divides n or 3 divides n. *Yes:* Both 2 and 3 divide n, so either divides n.

Which of these conditions are *sufficient*?
Solution: We discuss each case above.
If 3 or 9 divides n this does not imply 2 divides n, so i) and ii) are not sufficient.
If 12 divides n then $n = 12 \cdot q = 6 \cdot 2q$ for an integer q, so iii) is sufficient.
$n = 12$ implies 6 divides n, so iv) is sufficient.

We consider vi) before v) . 2 divides n implies $n = 2 \cdot p$. 3 divides $n = 2 \cdot p$ implies p is divisible by 3, or $p = 33 \cdot q$, so $n = 2 \cdot 3 \cdot q = 6 \cdot q$ is divisible by 6. So, v) is sufficient.

If 6 divides n^2, then both 2 and 3 divide n^2. This implies both 2 and 3 divide n, so by vi) 6 divides n and v) is sufficient.

vii) is not sufficient, as $n = 2$ is divisible by 2 or 3.
□

13. (*turn in Wednesday, September 5*)
Prove by induction on n that, for all positive integers n, $n^3 - n$ is divisible by 3.

Proof: We formulate the inductive statement first.

P(n): There exists integer q so that $n^3 - n = 3 \cdot q$.

We prove first the case P(1). For $n = 1$, $n^3 - n = 1^3 - 1 = 0 = 3 \cdot 0$.

Next, we assume that P(n) is true, and use this to prove the statement P(n+1). We calculate the LHS of P(n+1)

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + 3(n^2 + n)$$

Use the inductive hypothesis P(n) to write $n^3 - n = 3q$, then we have

$$(n+1)^3 - (n+1) = (n^3 - n) + 3(n^2 + n) = 3q + 3(n^2 + n) = 3(q + n^2 + n)$$

where $(q + n^2 + n)$ is an integer. This shows $(n+1)^3 - (n+1)$ is divisible by 3. So P(n+1) is also true, which completes the proof of the inductive step.

By the Principle of Induction, it follows that P(n) is true for all integers $n \geq 1$. $\square$