An Invitation to Higher Mathematics
Math 215, Fall Semester, 2001

Solutions to Problems & Exercises
Week 5 – September 17–21

19. Prove that

i) $\{x \in \mathbb{R} \mid x^2 + x - 2 = 0\} = \{-2, 1\}$

Proof: Set $A = \{x \in \mathbb{R} \mid x^2 + x - 2 = 0\}$ and $B = \{-2, 1\}$. Then

$$x \in A \iff x \in \mathbb{R} \ \& \ x^2 + x - 2 = 0 \iff x \in \mathbb{R} \ \& \ (x-1)(x+2) = 0$$

$$\iff x = 1 \textit{ or } x = -2 \iff x \in B \ \square$$

ii) $\{x \in \mathbb{R} \mid x^2 + x - 2 < 0\} = (-2, 1)$

Proof: Set $A = \{x \in \mathbb{R} \mid x^2 + x - 2 < 0\}$ and $B = (-2, 1)$. Then

$$x \in A \iff x \in \mathbb{R} \ \& \ x^2 + x - 2 < 0 \iff x \in \mathbb{R} \ \& \ (x-1)(x+2) < 0$$

$$\iff x \in \mathbb{R} \ \& \ ((x-1) < 0 \ \& \ (x+2) > 0) \textit{ or } ((x-1) > 0 \ \& \ (x+2) < 0)$$

$$\iff x \in \mathbb{R} \ \& \ (x < 1 \ \& \ x > -2) \textit{ or } (x > 1 \ \& \ x < -2)$$

$$\iff x \in \mathbb{R} \ \& \ -2 < x < 1 \iff x \in B \ \square$$

iii) $\{x \in \mathbb{R} \mid x^2 + x - 2 > 0\} = \{x \in \mathbb{R} \mid x < -2\} \cup \{x \in \mathbb{R} \mid x > 1\}$

Proof: Set $A = \{x \in \mathbb{R} \mid x^2 + x - 2 > 0\}$ and $B = (-\infty, -2) \cup (1, \infty)$. Then

$$x \in A \iff x \in \mathbb{R} \ \& \ x^2 + x - 2 > 0 \iff x \in \mathbb{R} \ \& \ (x-1)(x+2) > 0$$

$$\iff x \in \mathbb{R} \ \& \ ((x-1) < 0 \ \& \ (x+2) < 0) \textit{ or } ((x-1) > 0 \ \& \ (x+2) > 0)$$

$$\iff x \in \mathbb{R} \ \& \ (x < 1 \ \& \ x < -2) \textit{ or } (x > 1 \ \& \ x > -2)$$

$$\iff x \in \mathbb{R} \ \& \ (x < -2) \textit{ or } (x > 1) \iff x \in B \ \square$$

20. Find predicates which determine the following subsets of the set of integers $\mathbb{Z}$

i) $\{3\}$

Solution: $\{3\} = \{x \in \mathbb{Z} \mid x = 3\}$ $\square$

ii) $\{1, 2, 3\}$

Solution: $\{1, 2, 3\} = \{x \in \mathbb{Z} \mid 0 < x < 4\}$ $\square$

iii) $\{1, 3\}$ (*turn in this case on Monday, September 24*)

Solution: $\{1, 3\} = \{x \in \mathbb{Z} \mid 0 < x < 4 \;\&\; x \textit{ is odd}\}$ $\square$

Solution': $\{1, 3\} = \{x \in \mathbb{Z} \mid 0 < x \;\&\; x \textit{ is a divisor of } 3\}$ $\square$

21. (*turn in Monday, September 24*)

By using a truth table prove that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof: Make the truth table and compare the last two columns

$x \in A$	$x \in B$	$x \in C$	$x \in B \cup C$	$x \in A \cap (B \cup C)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

$x \in A$	$x \in B$	$x \in C$	$x \in A \cap B$	$x \in A \cap C$	$x \in (A \cap B) \cup (A \cap C)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

22. (*turn in Monday, September 24*)
By using a truth table prove that

$$A \subset B \iff A \cup B = B$$

$x \in A$	$x \in B$	$A \subset B$	$A \cup B$	$A \cup B = B$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	F	T

Compare the third and last columns to see they agree. $\square$

23. (*turn in Monday, September 24*)
Prove by contradiction or otherwise that

$$(A \cap B = A \cap C) \ \&\ (A \cup B = A \cup C) \iff B = C$$

Proof: Here is a direct proof.
If $B = C$ then both $A \cap B = A \cap C$ and $A \cup B = A \cup C$ are obvious.
Assume that $A \cap B = A \cap C$ and $A \cup B = A \cup C$ are given.
Observe $B = (B - A) \cup (B \cap A)$ and $C = (C - A) \cup (C \cap A)$, so it will suffice to show $B - A = C - A$. Use the two identities $A \cup B = A \cup (B - A)$ and $A \cup C = A \cup (C - A)$ then calculate

$$B - A = (A \cup (B - A)) - A = A \cup B - A = A \cup C - A = (A \cup (C - A)) = C - A$$

Proof: Here is another proof (by contradiction).
If $B = C$ then both $A \cap B = A \cap C$ and $A \cup B = A \cup C$ are obvious.
Suppose that $B \neq C$, then we must show that one of $A \cap B = A \cap C$ or $A \cup B = A \cup C$ is false.
Observe $B = (B - A) \cup (B \cap A)$ and $C = (C - A) \cup (C \cap A)$, so if $B - A = C - A$ and $B \cap A = C \cap A$ then $B = C$, so one of these must be false. If $B \cap A \neq C \cap A$ then this is what we needed to show. Otherwise, assume $B - A \neq C - A$. Then there are elements of $A \cup B = A \cup (B - A)$ not in $A \cup C = A \cup (C - A)$ so $A \cup B \neq A \cup C$. $\square$

24. (*turn in Monday, September 24*)
Using the fact that an implication is equivalent to its contra-positive, prove that, for subsets of a universal set U,

$$A \subset B \Longleftrightarrow B^c \subset A^c$$

Proof: Let P signify $x \in A$ and Q signify $x \in B$.
Then $\sim P$ signifies $x \in A^c$ and $\sim Q$ signifies $x \in B^c$.
The statement $A \subset B$ is equivalent to $P \Longrightarrow Q$
The statement $B^c \subset A^c$ is equivalent to $\sim Q \Longrightarrow \sim P$.
The equivalence of an implication with its contra-positive

$$(P \Longrightarrow Q) \Longleftrightarrow (\sim Q \Longrightarrow \sim P)$$

is then equivalent to

$$A \subset B \Longleftrightarrow B^c \subset A^c$$

Alternately, we can set up the truth table for the inclusions

$x \in A$	$x \in B$	$A \cup B$	A^c	B^c	$B^c \cup A^c$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

and the third and last columns agree. $\square$