An Invitation to Higher Mathematics
Math 215, Fall Semester, 2001

Problems & Exercises
Week 6 – September 24–28

"In mathematics you don't understand things, you just get used to them."
John von Neumann

25. Prove the following

$$(\exists q \in \mathbb{Z}, n = 2q + 1) \Longrightarrow \big(\exists p \in \mathbb{Z}, n^2 = 2p + 1\big)$$

Proof: For $n \in \mathbb{Z}$ we are given there is $q \in \mathbb{Z}$ with $n = 2q + 1$. Then

$$n^2 = (2q + 1)^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1$$

Set $p = 2q^2 + 2q$ and this shows $\exists p \in \mathbb{Z}, n^2 = 2p + 1$. $\square$

Just a remark: we have done this problem before, using a "different language". We proved "n is odd $\Longrightarrow n^2$ is odd", which is the same.

26. (*turn in Monday, October 1*)
Write the following statement in terms of quantifiers and prove it.
For integers a and b, if a and b are even then so is $a + b$.

Proof: First we write this using quantifiers, following the scheme of the problem above:

$$(\exists p \in \mathbb{Z}, n = 2p) \wedge (\exists q \in \mathbb{Z}, m = 2q) \Longrightarrow (\exists r \in \mathbb{Z}, nm = 2r)$$

The proof of this statement is now very quick.
Given $n = 2p$ and $m = 2q$, then $nm = 2p \cdot 2q = 4pq = 2(2pq)$.
Set $r = 2pq$. $\square$

27. (*turn in Monday, October 1*)
For sets A, B, C and D prove that

$$(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$$

and give an example to show that these sets are not always equal.

Proof 1: We prove this using a sequence of equivalent statements:

$$\begin{aligned}(x,y) \in (A\times B)\cup(C\times D)&\\ &\iff [(x,y)\in(A\times B)]\vee[(x,y)\in(C\times D)]\\ &\iff [(x\in A)\wedge(y\in B)]\vee[(x\in C)\wedge(y\in D)]\\ &\implies [(x\in A)\vee(x\in C)]\wedge[(y\in B)\vee(y\in D)]\\ &\iff (x,y)\in(A\cup C)\times(B\cup D)\end{aligned}$$

Proof 2: Use the distributive property of set products to expand the product of the two sets

$$(A\cup C)\times(B\cup D) = A\times B\ \cup\ A\times D\ \cup\ C\times B\ \cup\ C\times D$$

The union of the two products $(A\times B)\cup(C\times D)$ is a subset of this. □

The second proof shows "what is going on" but unfortunately, we haven't proved the "distributive property of set products" yet, so quoting it is not correct. The most elegant proof would be to prove this general rule following the steps of the first proof (replace the $\implies$ line with an $\iff$ line, and change the last line) then apply it like above.

For an example where the two sets are not equal, take $A=\{a\}, B=\{1\}, C=\{b\}, D=\{2\}$. Then $(A\times B)\cup(C\times D)=\{(a,1),(b,2)\}$ while $(A\cup C)\times(B\cup D)=\{(a,1),(a,2),(b,1),(b,2)\}$.

28. (*turn in Monday, October 1*)
Define functions $f, g\colon \mathbb{R}\to\mathbb{R}$ by $f(x)=x^3$ and $g(x)=1-x$.
a) Find the functions
i) $f\circ f$ *Solution:* $f\circ f(x)=f(x^3)=(x^3)^3=x^9$.
ii) $f\circ g$ *Solution:* $f\circ g(x)=f(1-x)=(1-x)^3=1-3x+3x^2-x^3$.
iii) $g\circ f$ *Solution:* $g\circ f(x)=g(x^3)=1-x^3$.
iv) $g\circ g$ *Solution:* $g\circ g(x)=g(1-x)=1-(1-x)=x$.

b) List the elements of the set $\{x\in\mathbb{R}\mid fg(x)=gf(x)\}$.
Solution:

$$\begin{aligned}\{x\in\mathbb{R}\mid f\circ g(x)=g\circ f(x)\}&=\{x\in\mathbb{R}\mid 1-3x+3x^2-x^3=1-x^3\}\\ &=\{x\in\mathbb{R}\mid -3x+3x^2=0\}\\ &=\{x\in\mathbb{R}\mid x(x-1)=0\}\\ &=\{0,1\}\end{aligned}$$

29. (*turn in Monday, October 1*)
Define the composition of the function $f: X \to Y$ and the function $g: Y \to Z$ to be the function $g \circ f: X \to Z$ with $g \circ f(x) = g(f(x))$ for all $x \in X$. Prove that:
a) If f is injective and g is injective, then $g \circ f$ is injective.
b) If f is surjective and g is surjective, then $g \circ f$ is surjective.

Proof of a): By the assumption f is injective,

$$x \neq y \Longrightarrow f(x) \neq f(y)$$

By the assumption g is injective,

$$f(x) \neq f(y) \Longrightarrow g(f(x)) \neq g(f(y))$$

Combining these two statements gives

$$x \neq y \Longrightarrow g \circ f(x) \neq g \circ f(y)$$

This is the statement that $g \circ f$ is injective. $\square$

Proof of b): By the assumption g is surjective,

$$\forall z \in Z, \exists y \in Y, z = g(y)$$

By the assumption f is surjective,

$$\forall y \in Y, \exists x \in X, y = f(x)$$

Combining these two statements gives

$$\forall z \in Z, \exists y \in Y, z = g(y) \text{ and } \exists x \in X, y = f(x) \Longrightarrow z = g(f(x))$$

This is the statement that $g \circ f$ is surjective. $\square$