An Invitation to Higher Mathematics
Math 215, Fall Semester, 2001

Solutions to Problems & Exercises
Week 8 – October 8 –12

30. (*turn in Monday, October 15*)
Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$.

a) What is the cardinality of the cartesian product $A \times B$?

Solution: $3 \times 4 = 12$.

b) Give an explicit bijection $f : \mathbb{N}_n \to A \times B$ where $n = |A \times B|$.

Solution: Order the set and count them off. For example,

$$\begin{aligned} A \times B = &\{(a,1),(a,2),(a,3),(a,4),\\ &(b,1),(b,2),(b,3),(b,4),\\ &(c,1),(c,2),(c,3),(c,4)\} \end{aligned}$$

Set $f(1) = (a, 1)$, $f(2) = (a, 2)$ and so on.

c) Give a second bijection $g : \mathbb{N}_n \to A \times B$ distinct from your answer to part a.

*Solution:*Try $g(1) = (a, 2)$, $g(2) = (a, 1)$ and $g(i) = f(i)$ for $3 \leq i \leq 12$.

d) How many *possible* bijections are there between the sets $\mathbb{N}_n$ and $A \times B$?

Solution: There are 12 factorial ways to map 12 items to 12 items, or 479,001,600 ways.

31. (*turn in Monday, October 15*)
Of the 170 students who took all the first year core modules last year, 124 liked Reasoning, 124 liked Algebra, 124 liked Calculus, 10 liked only Reasoning, nobody liked only Algebra, 4 liked only Calculus and 2 liked none of the modules. How many students liked all three core modules?

Solution: Introduce variables for all the data:

$$
\begin{aligned}
|R| =& 124 \\
|A| =& 124 \\
|C| =& 124 \\
|R - (A \cup C)| =& 10 \\
|A - (R \cup C)| =& 0 \\
|C - (R \cup A)| =& 4 \\
|U - (R \cup A \cup C)| =& 2
\end{aligned}
$$

Introduce the other unknown variables

$$
\begin{aligned}
|(R \cap A) - C| =& x \\
|(A \cap C) - R| =& y \\
|(C \cap R) - A| =& z \\
|(R \cap A \cap C| =& w
\end{aligned}
$$

Adding up the totals for each set R, A, C and U gives

$$
\begin{aligned}
|R| = 10 + x + z + w =& 124 \\
|A| = 0 + x + y + w =& 124 \\
|C| = 4 + y + z + w =& 124 \\
|U| = 10 + 0 + 4 + x + y + z + w + 2 =& 170
\end{aligned}
$$

Simplifying gives

$$
\begin{aligned}
x + z + w =& 114 \\
x + y + w =& 124 \\
y + z + w =& 120 \\
x + y + z + w =& 154
\end{aligned}
$$

Subtracting each of the first three lines from the last gives

$$
\begin{aligned}
y =& 40 \\
z =& 30 \\
x =& 34 \\
w =& 50
\end{aligned}
$$

So the asnwer is $w = |R \cap A \cap C| = 50$.

32. (*turn in Monday, October 15*)
Let $A = \{1, 2, 3, 4\}$. List all of the elements of its *power set* $\mathcal{P}(A)$.
Solution: $\mathcal{P}(A)$ is just the collection of all subsets of A, so we have:

$$\begin{aligned}\mathcal{P}(A) = \{&\emptyset,\\ &\{1\}, \{2\}, \{3\}, \{4\},\\ &\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\},\\ &\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\},\\ &\{1,2,3,4\}\}\end{aligned}$$

Observe that $|\mathcal{P}(A)| = 1 + 4 + 6 + 4 + 1 = 16 = 2^4 = 2^{|A|}$

33. (*turn in Monday, October 15*)
For the set $B = \{\alpha, \beta, \gamma\}$.
a) Find all of the functions from B to the set $\{0, 1\}$. Write down the functions explictly using a table with rows for the elements of B, and a column for each function.

Solution:

x	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
α	0	1	0	0	1	1	0	1
β	0	0	1	0	1	0	1	1
γ	0	0	0	1	0	1	1	1

b) For each of the functions f you list in part a) write down the set $B_f = f^{-1}(1)$. What can you say about the collection of sets you obtain?

Solution: The sets $B_i = f_i^{-1}(1)$ for the functions in the list above correspond to all of the subsets of B.