An Invitation to Higher Mathematics
Math 215, Fall Semester, 2001

Solutions to Problems & Exercises
Week 9 – October 15 –19

34. (*turn in Monday, October 22*)
For $f\colon X \to Y$ and $g\colon Y \to Z$ show that:
a) if $g \circ f$ is injective, then f is injective.
b) if $g \circ f$ is surjective, then g is surjective.

Proof of a): It is given that $g \circ f$ is injective:

$$(P) \qquad \forall x, y \in X, x \neq y \Longrightarrow g(f(x)) \neq g(f(y))$$

We need to prove that f is injective:

$$(Q) \qquad \forall x, y \in X, x \neq y \Longrightarrow f(x) \neq f(y)$$

The easiest method is to prove the contrapositive: $\sim Q \Longrightarrow \sim P$.
Assume f is not injective, then there exists $x, y \in X, x \neq y$ with $f(x) = f(y)$. Apply g to both sides to get $g(f(x)) = g(f(y))$. This shows that $g \circ f$ is not injective, which is $\sim P$. □

Proof of b): It is given that $g \circ f$ is surjective:

$$\forall z \in Z, \exists x \in X, z = g(f(x))$$

We need to prove that g is surjective:

$$\forall z \in Z, \exists y \in Y, z = g(y)$$

The direct proof is easiest. Given $z \in Z$, by assumption there exists $x \in X$ with $z = g(f(x))$. For this x let $y = f(x)$. The $g(y) = g(f(x)) = z$, so for this y we have $z = g \circ f(x)$. □

35. (*turn in Monday, October 22*)
For $f: X \to Y$ and $g: Y \to X$, use problem 34 to prove that if $g \circ f$ is injective and $f \circ g$ is surjective, then f is bijective.

Proof: Suppose that $g \circ f$ is injective, then by 34) f injective.
Suppose that $f \circ g$ is surjective, then by 34) f surjective.
With the assumption that $g \circ f$ is injective and $f \circ g$ is surjective we conclude f is both injective and surjective, or f is bijective. □

36. (*turn in Monday, October 22*)
For $f: X \to Y$ and $g: Y \to X$, if $g \circ f$ and $f \circ g$ are both bijective, show that f and g are both bijective.

Proof: Assume that $f \circ g$ and $g \circ f$ are both bijective.
First, $f \circ g$ is injective and $g \circ f$ is surjective, so by problem 35) f is bijective.
Second, $g \circ f$ is injective and $f \circ g$ is surjective, so by problem 35) g is bijective. □

37. (*turn in Monday, October 22*)
The functions $f: X \to Y$ and $g: Y \to X$ are inverses of each other if $\forall x \in X,\ g(f(x)) = x$, and $\forall y \in Y,\ f(g(y)) = y$. Use problem 36 to prove that f and g are both bijections.

Proof: Assume that f and g are inverses of each other. Then the compositions $f \circ g = id_Y$ and $g \circ f = id_X$. The identity map of any set is always injective and surjective, so in particular the maps $id_x: X \to X$ and $id_Y: Y \to Y$ are both bijections. Then by problem 36) both f and g are bijections. □

38. (*turn in Monday, October 22*)
Given five points in the plane with integer coordinates, prove that the midpoint between at least one of pair of the points has *integer* coordinates.

Proof: Each point $(m, n) \in \mathbb{Z}^2$ has an even/odd parity associated to each coordinate m and n. Let $P = \{(e, e), (e, o), (o, e), (o, o)\}$ be the set of parities. Define a map $p\colon \mathbb{Z}^2 \to P$ by setting

$$p(m, n) = (parity\ m, parity\ n)$$

For example, $p(2, 3) = (e, o)$.

Let $\mathcal{S} \subset \mathbb{Z}^2$ be a set of 5 distinct points with integer coordinates. The restriction of the map $p\colon \mathcal{S} \to P$ maps a set of 5 elements to a set of 4 elements, so by the "pigeon hole principle" the map p cannot be injective. This means there must be distinct pairs of points $(m, n) \in \mathcal{S}$ and $(r, s) \in \mathcal{S}$ with $p(m, n) = p(r, s)$. We write out what this means in the four possible cases:

If m is odd, then r must also be odd; while if m is even, then r must also be even. In both cases, the sum $m + r$ is even, so divisible by 2.

If n is odd, then s must also be odd; while if n is even, then s must also be even. In both cases, the sum $n + s$ is even, so divisible by 2.

The midpoint of (m, n) and (r, s) is the point $(\frac{m+r}{2}, \frac{n+s}{2})$ which has integer coordinates. □

39. (*turn in Monday, October 22*)
Let $S \subset \{1, 2, 3, \ldots, 2n\}$ where S has $n + 1$ elements. Show that S contains two numbers such that one divides the other. Recall that $n|m$ means there is some integer p so that $m = p \cdot n$.

Hint: Any number m can be written uniquely as an odd number times a power of 2, or $m = (2k - 1) \cdot 2^j$. Then consider the function $f\colon S \to \{1, 2, 3, \ldots, n\}$ defined by $f(m) = k$.

Proof: Each positive integer m can be divided by 2 as many times as needed until the remainder is a positive integer. For example, $14 = 2 \cdot 7$ while $28 = 4 \cdot 7 = 2^2 \cdot 7$. A positive integer can be written as $(2k - 1)$ where k is a positive integer. Combine the division with this way of writing odd integers to get every positive integer m can be written as $m = (2k - 1) \cdot 2^j$.

Use this to define a function from the integers to the integers

$$f\colon \{1, 2, 3, \dots\} \to \{1, 2, 3, \dots\} \; by \; f(m) = k, \; if \; m = (2k - 1) \cdot 2^j$$

For example, $f(14) = 4$ as $14 = 7 \cdot 2 = (2 \cdot 4 - 1) \cdot 2$. Also, $f(28) = 4$ as $28 = 7 \cdot 4 = (2 \cdot 4 - 1) \cdot 2$. Two more examples are $f(100) = 13$ and $f(8) = 1$ (check these!)

Back to the problem: given the set $S \subset \{1, 2, 3, \dots, 2n\}$ we restrict the function $f\colon S \to \{1, 2, 3, \dots\}$. The key point is to figure out the maximum value for f on the set $\{1, 2, 3, \dots, 2n\}$ as this will also be a maximum value for f on S.

When $x = 2k - 1$ is odd, then $f(2k - 1) = k$. So $2k - 1 \leq 2n$ implies $k \leq n + 1/2$ or $k \leq n$ as both k and n are integers.

When x is even then $x = (2k - 1) \cdot 2^j$ where $2^j > 1$. If $x \leq 2n$ then $2^j > 1$ implies $(2k - 1) \leq 2n/2^j \leq 2n$ so again $k \leq n$.

Combining the odd and even cases, we have shown the value of $f(x)$ is at most n when $x \leq n$, so $f\colon \{1, 2, 3, \dots, 2n\} \to \{1, 2, \dots, n\}$.

The restriction of f to the set S maps a set with $n + 1$ elements to a set with n elements, so $f\colon S \to \{1, 2, \dots, n\}$ cannot be injective. Therefore, there must be distinct integers $x, y \in S$ so that $f(x) = f(y)$. Let $k = f(x) = f(y)$ be the common value. Then $x = (2k - 1) \cdot 2^j$ and $y = (2k - 1) \cdot 2^\ell$ for some integers $j, \ell \geq 0$.

If $j \geq \ell$ then

$$\frac{x}{y} = \frac{(2k - 1) \cdot 2^j}{(2k - 1) \cdot 2^\ell} = 2^{j - \ell}$$

is an integer so x divides y. Otherwise, $\ell < j$

$$\frac{y}{x} = \frac{(2k - 1) \cdot 2^\ell}{(2k - 1) \cdot 2^j} = 2^{\ell - j}$$

is an integer so y divides x. $\square$