An Invitation to Higher Mathematics
Math 215, Fall Semester, 2001

Solutions to Problems & Exercises
Week 10 – October 22–26

40. (*turn in Wednesday, October 31*)
Show that the set of all positive multiple of 3 is a countable set. (Your answer should be a bijection from $\mathbb{N}$ to this subset of the positive integers.)

Proof: The set of all positive multiples of 3 is the set

$$\mathcal{S} = \{3, 6, 9, \dots\} = \{3 \cdot n \mid n \in \mathbb{N}\}$$

The problem asks to show $\mathcal{S}$ is countable, which means show there is a bijection from $\mathbb{N}$ to $\mathcal{S}$. We construct the map, then show it is a bijection.

Define a map $f\colon \mathbb{N} \to \mathcal{S}$ by setting $f(n) = 3n$.

f is surjective, as for every element $y \in \mathcal{S}$ y is divisible by 3, so there is some $n \in \mathbb{N}$ with $y = 3n$. Then $f(n) = y$.

f is injective, as $f(n) = f(m)$ implies $3n = 3m$, or $n = m$. □

41. (*turn in Wednesday, October 31*)
Let A be a countably infinite set with bijection $g\colon \mathbb{N} \to A$. Suppose that B is a finite set with $A \cap B = \emptyset$. Show that $A \cup B$ is countably infinite. (Your answer should use the function g and a counting of the set B to produce a bijection from $\mathbb{N}$ to $A \cup B$.)

Proof: Use the bijection $g\colon \mathbb{N} \to A$ to set $a_n = g(n)$. Then

$$A = \{a_1, a_2, a_3, \dots\}$$

The set B is finite, so $|B| = m$ for some $m \in \mathbb{N}$ and there is a bijection $f\colon \mathbb{N}_m \to B$. Use the bijection f to set $b_n = f(n)$. Then

$$B = \{b_1, b_2, \dots, b_m\}$$

Define a map $h\colon \mathbb{N} \to A \cup B$ by setting $h(n) = b_n$ for $1 \leq n \leq m$, and $h(n) = a_{n-m}$ for $n > m$.

h is surjective by construction – the image of the first m integers list all of the elements of B, and the image of the rest of the integers, from $m + 1$ on, lists the elements of A.

We must also show h is injective. Consider $n \neq n'$. If both $n, n' \leq m$ then $h(n) = h(n')$ means $b_n = b_{n'}$ which is impossible. If both $n, n' > m$ then $h(n) = h(n')$ means $a_{n-m} = a_{n'-m}$ which is also impossible. If $n \leq m$ and $n' > m$ then $h(n) = h(n')$ means $b_n = a_{n'-m}$ which is impossible as $A \cap B = \emptyset$. The case $n > m$ and $n' \leq m$ is the same. □

42. (*turn in Wednesday, October 31*)
Let A and B be a countable sets, which need not be disjoint. Show that $A \cup B$ is countable.

Proof: The problem will be broken into three cases. First, if A and B are both finite sets, then $A \cup B$ is finite so is countable.

If A in infinite and B is finite (or vice-versa,) then we can reduce this problem to problem 41. Replace B with the subset $C = B - A \subset B$ of elements of B not in A, then $A \cup B = A \cup C$, C is finite and $A \cap C = \emptyset$, so $A \cup C$ is countable by problem 41.

The third case is the new part of this problem. Assume A and B are countably infinite. There are bijections from $\mathbb{N}$ to each, so we can write the sets as

$$A = \{a_1, a_2, a_3, \dots\}$$

$$B = \{b_1, b_2, b_3, \dots\}$$

Define $f \colon \mathbb{N} \to A \cup B$ by $f(2n - 1) = a_n$ and $f(2n) = b_n$.

The map f is onto $A \cup B$ by construction – the odd numbers are sent to the elements of A, and the even numbers are sent to the elements of B. If $A \cap B = \emptyset$ then an argument like in problem 41 shows that f is injective. But $A \cap B$ may not be empty, so we can't use this argument.

We fix this problem in two steps. First, we chose a subset $\mathcal{S} \subset \mathbb{N}$ so that the restriction $f \colon \mathcal{S} \to A \cup B$ is a bijection. Then, since $\mathcal{S}$ is a subset of a countable set, it is also countable, say by a bijection $g \colon \mathbb{N} \to \mathcal{S}$. The composition $f \circ g \colon \mathbb{N} \to \mathcal{S} \to A \cup B$ is a bijection, so $A \cup B$ is countable.

Now we define $\mathcal{S}$. First, $1 \in \mathcal{S}$. Next, $2 \in \mathcal{S}$ if $f(2) \neq f(1)$ (this means $b_1 \neq a_1$.) Otherwise, don't include 2. After that, $3 \in \mathcal{S}$ if $f(3) \neq f(1)$ and $f(3) \neq f(2)$. Otherwise, don't include 3.

The pattern is then clear, and proceeds inductively: $n \in \mathcal{S}$ if $f(n) \neq f(i)$ for all $1 \leq i < n$. In other words,

$$n \in \mathcal{S} \iff f(n) \notin \{f(1), f(2), \dots, f(n-1)\}$$

We show that f restricted to $\mathcal{S}$ is surjective. Given any $y \in A \cup B$ we know $f \colon \mathbb{N} \to A \cup B$ is surjective, so the set $f^{-1}(y) \subset \mathbb{N}$ is not empty.

Every subset of $\mathcal{N}$ has a smallest element. Let $n \in f^{-1}(y)$ be the least element of this set. If $f(n) = f(i)$ for some $i < n$ then $i \in f^{-1}(y)$ so n wasn't the least element. This shows $f(n) \neq f(i)$ for $i < n$, so $n \in \mathcal{S}$.

This shows $y = f(n)$ with $n \in \mathcal{S}$, so y is in the image of $\mathcal{S}$. As $y \in A \cup B$ was arbitrary, this shows $f\mathcal{S} \to A \cup B$ is a surjection.

We show that f restricted to $\mathcal{S}$ is injective. The proof is by contradiction. Suppose that $1 \leq n < n'$ are integers with $n, n' \in \mathcal{S}$ and $f(n') = f(n)$. This implies

$$f(n') \in \{f(1), f(2), \ldots, f(n), \ldots, f(n'-1)\}$$

which contradicts the definition of $\mathcal{S}$. So $f(n') = f(n)$ is impossible. □

43. (*turn in Wednesday, October 31*)
Show that the product $\mathbb{N} \times \mathbb{N}$ is countable. (Your answer should be a bijection $f\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

Proof: There are lots of ways to build a bijection $f\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ – we construct a bijection using the idea from the solution of problem 39.

Each integer $n \in \mathbb{N}$ can be written in the form $n = (2k-1) \cdot 2^i$ where $k \geq 1$ and $i \geq 0$. Use this to define $f(n) = (k, i+1)$.

We show f is a bijection by writing down its inverse $g\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Define $g(k, \ell) = (2k-1) \cdot 2^{\ell-1}$. Check:

$$g \circ f(n) = g(f((2k-1) \cdot 2^i)) = g(k, i+1) = (2k-1) \cdot 2^{i+1-1} = n$$

$$f \circ g(k, \ell) = f((2k-1) \cdot 2^{\ell-1}) = (k, \ell - 1 + 1) = (k, \ell)$$

Then by problem 37, both f and g are bijections. □