An Invitation to Higher Mathematics
Math 215, Fall Semester, 2001

Solutions to Problems & Exercises
Week 11 – October 29–November 2

44. (*turn in Monday, November 5*)
Prove that for sets A, B, C

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Proof: The idea is to use the definition of the Cartesian product of two sets, along with the definition of union as the "or" of inclusion in either B or C.

$$\begin{aligned}(x,y) \in A \times (B \cup C) &\iff (x \in A) \ \text{ and } \ (y \in B \cup C) \\ &\iff (x \in A) \ \text{ and } \ (y \in B \text{ or } y \in C) \\ &\iff (x \in A \text{ and } y \in B) \ \text{ or } \ (x \in A \text{ and } y \in C) \\ &\iff (x,y) \in A \times B \text{ or } (x,y) \in A \times C \\ &\iff (x,y) \in (A \times B) \cup (A \times C)\end{aligned}$$

The proof can also be written out using cases like for the solution of problem 3 on Hour Exam 1. □

45. (*turn in Monday, November 5*)
Write down the first *ten* rows of Pascal's Triangle. Use this to find the coefficient of a^7b^3 in $(a+b)^{10}$.

Solution: Ooops, 11 rows. The 11^{th} row is

1 10 45 120 210 252 210 120 45 10 1

The coefficient of a^7b^3 is then the 7^{th} entry, or 120, starting the counting at 0. It is almost as easy to write down the full expansion

$$\begin{aligned}(a+b)^{10} = 1a^{10} &+ 10a^9b^1 + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 \\ &+ 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10a^1b^9 + 1b^{10}\end{aligned}$$

46. (*turn in Monday, November 5*)
Find the coefficient of x^4 in $(x+2)^7$.

Solution: The long answer is to use the binomial theorem to expand out the entire expression, which we do just to see the answer:

$$
\begin{aligned}
&(x+2)^7 \\
&\quad = x^7 + 7x^6 2^1 + 21x^5 2^2 + 35x^4 2^3 + 35x^3 2^4 + 21x^2 2^5 + 7x^1 2^6 + 2^7 \\
&\quad = x^7 + 14x^6 + 84x^5 + 280x^4 + 560x^3 + 672x^2 + 448x + 128
\end{aligned}
$$

The short answer is that the x^4 term of the expansion is $35x^4 2^3 = 280x^4$, so the answer is 280.

47. (*turn in Monday, November 5*)
Three people each select a main dish from a menu of five items. How many choices are possible
(i) if we record who selected which dish (as the waiter should)
(ii) if we ignore who selected which dish (as the chef could)?

Solution: i) There are 3 people, and each can order one of 5 dishes. This can be thought of as a function from a 3 element set (the customers) to a 5 element set (the dishes). Either way, the number of choices is $5 \times 5 \times 5 = 125$.

ii) This problem takes some thought. The chef sees an order for 3 dishes. How many possible orders can he encounter? We count them by looking at three cases:

a - all 3 dishes are the same. Then the only thing needed to know is which dish got ordered 3 times. So there are 5 choices.
b - 2 dishes are the same, and the third is distinct. Then the chef needs to know which is the distinct dish (5 choices) and which is the double dish (4 choices). So the total number of choices is 20.
c - all 3 dishes are distinct. Then the number is the same as choosing 3 things from 5, or 10.

Add up the three cases to get the answer, $5 + 20 + 10 = 35$.

48. (*turn in when you solve it!*)
Prove that the product of any n consecutive positive integers is divisible by $n!$

Proof: Look at some examples:

$17 \cdot 18 \cdot 19$ is divisible by $3! = 3 \cdot 2 \cdot 1 = 6$. True? why?

$17 \cdot 18 \cdot 19 \cdot 20 \cdot 21$ is divisible by $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. True? why?

49. (*turn in when you solve it!*)
Use the pigeonhole principle to prove that, given ten distinct positive integers less than 107, there exists two disjoint subsets with the same sum.

Proof: We are given distinct integers $1 \leq n_1 < n_2 < \cdots < n_{10} < 107$.

How many possible ways can you choose two disjoint subsets?
(Note the subsets don't have to be all of the 10 integers!)

How large is the possible sum of the integers in one of the subsets?