Math 215: "proof course" **Notes on problem 10 = 4.7**

posted Wednesday 30 January 2002

Recall we mentioned at least three methods for doing this problem:

–by cases; this is fairly lengthy because of subcases
–by contradiction (as done by Gary Spencer in class, Mon 28jan02)
–by use of 4.6, as mentioned briefly in class Fri 25jan02.

One student asked for more details of the last method. This is the method appearing in the back of the book. But since the book won't be in the bookstore for another week or two, today I'll post those details:

To prove: for all real numbers a and b,

$$|a + b| \leq |a| + |b|.$$

Give a necessary and sufficient condition for equality.

The proof in the back of the book (I've expanded a bit) says:

Notice that $||a| + |b|| = |a| + |b|$, since $|a| \geq 0$ and $|b| \geq 0$.

We will "work backwards", as in section 3.2; since we want to get a necessary and sufficient condition, we will try to make each step reversible—that is, "if and only if":

Using the previous question (that is problem 4.6—combining the two parts, we get $|a| \leq |b|$ if and only if $a^2 \leq b^2$), with $a + b$ and $|a| + |b| = ||a| + |b||$ in the roles of "a" and "b", we see that the desired conclusion above holds if and only if:

$$(a + b)^2 \leq (|a| + |b|)^2.$$

This in turn holds if and only if, expanding out both squares:

$$a^2 + 2ab + b^2 \leq |a|^2 + 2|a||b| + |b|^2.$$

Now by 1.5, $|a|^2 = a^2$ and $|b|^2 = b^2$. So the above holds if and only if, subtracting off these equal terms:

$$2ab \leq 2|a||b|.$$

Of course the multiplication law for inequalities (or the same law for equations, if this is an equality), we can multiply both sides by the positive number $\frac{1}{2}$ to get

$$ab \leq |a||b|.$$

Now this statement is always true, as we can see by using 4.4.3: If a and b have the same sign, then a is positive, and the above statement is an equality. If a and b have opposite signs, so that ab is negative and hence $ab = -|a||b|$, then the above statement is either a strict inequality ($<$), or an equality when one of a and b is 0.

This completes the proof of the original statement.

Furthermore if in all steps we replace $\leq$ by $=$, then all steps are still true and reversible—except possibly the last. However we noticed that if a and b have the same sign (that is, $ab \geq 0$), then the last step is an equality. So $ab \geq 0$ is a necessary and sufficient condition for equality.

Q.E.D.