Math 310: Hour Exam 1 
Prof. S. Smith: Mon 28 Sept 1998

(Solutions)

**Problem 1:** (a) Using Gauss-Jordan, find the row-reduced echelon form of the following augmented matrix. (INDICATE the row operations you use).

\[
(A|b) = \begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 0 & 1 & 2 & | & 3 \\ 2 & 3 & 4 & | & 9 \end{pmatrix}
\]

\[
A^3_{2 \times 3} \rightarrow \begin{pmatrix} 1 & 2 & 3 & | & 6 \\ 0 & 1 & 2 & | & 3 \\ 0 & -1 & -2 & | & -3 \end{pmatrix} A^3_{1 \times 2} \rightarrow A^2_{3 \times 2} \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 1 & 2 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}
\]

(b) Then give the solutions of the corresponding linear system \(Ax = b\).

So solutions are \((r, 3 - 2r, r)\) for free variable \(x_3 = r\).

**Problem 2:** (a) What elementary row OPERATION will change

\[
A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{to} \quad B = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}
\]

\[A^-2_{2 \times 1} : \text{add } -2 \text{ times 1st row to 2nd.}\]

(b) What elementary row MATRIX \(E\) will, by left multiplication, perform the same operation? (that is, \(EA = B\))

\[
E = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} : \text{(obtained by doing that operation to the identity matrix)}
\]

**Problem 3:** (a) Find the inverse (any method) of:

\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}
\]

Using \((A|I)\) method: use \(A^{-1}_{2 \times 1}, A^{-1}_{3 \times 1}\) to clear first column; then \(A^{-1}_{3 \times 2}\) to clear second column.

Get inverse:

\[
\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}
\]

(b) Give the \(LU\)-decomposition of

\[
A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix};
\]

that is, find lower-triangular \(L\) and upper-triangular \(U\), so that \(A = LU\).

Apply \(A^{-1}_{2 \times 1}\) to get \(U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\). So \(L\) from inverse operation \(A^{-1}_{2 \times 1}\) is \(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\)
**Problem 4:** (a) Find the determinant of

\[ A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}. \]

*Top row:* \(1(1.1 - 0.1) - 1(0.1 - 1.1) + 0 = (1) - (-1) = 2.\)

(b) Use Cramer’s rule to solve

\[
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}
\]

\[
\det(A) = 1.4 - 3.2 = -2, \text{ so } x_1 = -\frac{1}{2}(4.4 - 10.2) = 2 \text{ and } x_2 = \frac{1}{2}(1.10 - 3.4) = 1.
\]

**Problem 5:** (a) Is \((1, 0, 1)\) in the span of \((1, 1, 1)\) and \((1, 2, 1)\)?

Either give coefficients in a linear combination, or explain why it is not possible.

*Yes:* Set up augmented matrix \((A|b) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}.\)

Row-reduction quickly produces coefficients 2 and -1.

(b) Let \(V\) be the space of \(2 \times 2\) matrices, and \(W\) the subset of diagonal matrices. Show that \(W\) is a subspace of \(V\).

\(A \in W\) says diagonal, meaning 0 off the diagonal, so \(A\) has form \(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.\)

Similarly if \(B \in W\) it has form \(\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\).

Is \(A + B \in W\)? It is \(\begin{pmatrix} a+c & 0 \\ 0 & b+d \end{pmatrix}\), also diagonal, so yes.

For scalar \(r\), is \(rA \in W\)? It is \(\begin{pmatrix} ra & 0 \\ 0 & rb \end{pmatrix}\), also diagonal, so yes.

We see “yes”, \(W\) is a subspace of \(V\).