Math 310: Hour Exam 1 (Solutions)
Prof. S. Smith: Fri 13 October 2000

Problem 1: (a) Using either Gaussian or Gauss-Jordan elimination, find all solutions of the linear equation system $Ax = b$ determined by the following augmented matrix: $(A|b) = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 2 & -1 & 1 & 3 \\ 4 & -3 & 1 & 0 \end{pmatrix}$.

SHOW the steps of the method you use.

$$
Use \ text{ row operations (Gauss-Jordan)} \begin{pmatrix} \mathbb{R}_1 \\ \mathbb{R}_2 \\ \mathbb{R}_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 1 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -1 \end{pmatrix}
$$

The bottom row says $0 = 1$, so there are NO solutions.

(b) Which two row operations will bring the matrix $A = \begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{pmatrix}$ into row-reduced echelon form $U$? Give elementary row matrices accomplishing those operations by left multiplication; this, give $E_1$ and $E_2$ such that $E_2E_1A = U$.

Row operations: $A_1^{4 \times 2}$ and $A_3^{-5 \times 2}$. Matrices: $E_1 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}$

Problem 2: (a) Is the matrix $E_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$ row-equivalent to the identity matrix $I_3$?

Say why/why not.

No: for example, $det(A) = 0$, so $A$ is singular; so the row-reduced echelon form of $A$ has a row of zeros (or, only 2 pivots), and hence cannot be the identity $I_3$.

(b) Given the augmented matrix $(A|b) = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \end{pmatrix}$, find the inverse (by any method) of the coefficient matrix $A$; use $A^{-1}$ to give a solution of the corresponding linear system $Ax = b$.

By adjoint method, $A^{-1} = \begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$, so $x = A^{-1}b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Problem 3: (a) Find the determinant of the product $AB$, where

$$
A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 5 & 4 \\ 0 & 6 & 5 \\ 0 & 0 & 1 \end{pmatrix}.
$$

SHOW the steps you used (“calculator” is not sufficient for credit here).

As $A$ is triangular, $det(A)$ is the product down the diagonal, namely $1.2.1 = 2$.

Similarly $det(B) = 1.6.1 = 6$. So $det(AB) = det(A)det(B) = 2.6 = 12$.

(b) For $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$, find the adjoint $adj(A)$, and then the inverse $A^{-1}$.

We need the transpose of the matrix of cofactors, namely $adj(A) = \begin{pmatrix} 5 & -4 \\ -3 & 2 \end{pmatrix}$.

Then $det(A) = 2.5 - 3.4 = -2$, $A^{-1} = \frac{1}{det(A)}adj(A) = -\frac{1}{2}\begin{pmatrix} 5 & -4 \\ -3 & 2 \end{pmatrix}$. 

1
Problem 4: (a) In the space $\mathbb{R}^{2 \times 2}$ of all 2x2 matrices, show that the set of all upper-triangular matrices forms a subspace. What is the dimension of this subspace.

(+) Add two general upper-triangular matrices: \[
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} a+d & b+e \\ 0 & c+f \end{pmatrix};
\]
so the sum is also upper-triangular.

(sc.mult.) Multiply a scalar times a general upper-triangular matrix: \[
f \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} fa & fb \\ 0 & fc \end{pmatrix};
\]
so the product is also upper-triangular.

Dimension: the “free variables” are $a, b, c$ above, so the dimension is 3.

(b) What is the dimension of the span of the columns of the matrix $A = \begin{pmatrix} 1 & 1 & 3 & 0 \\ 0 & 2 & 4 & 2 \\ 1 & 3 & 7 & 2 \end{pmatrix}$?

(Explain how you know this is the dimension).

The dimension is 2. One way: The row-reduced echelon form of $A$ is \[
\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Only 2 pivots, so $\text{dim(col.space)} = 2$. The first two columns give one basis for the col.space).

Problem 5: (a) Find a basis for the row space of the matrix $A = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 2 & 3 & 1 & -4 \\ 3 & 4 & 3 & -5 \end{pmatrix}$.

(Say why you know your answer gives a basis for the row space).

The row-reduced echelon form of $A$ is \[
\begin{pmatrix} 1 & 0 & 5 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Two pivots—so first two rows of rref give one basis for row space. Or, first two rows of $A$.

(b) Find the matrix of transition from the “old” basis given by the standard basis of $\mathbb{R}^2$ (namely $(1,0)^T$ and $(0,1)^T$) to the “new” basis given by $(3,5)^T$ and $(1,2)^T$.

One way: The matrix $[\text{new}]_{old}$ is given by \[
\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix},
\]

so the transition matrix $[\text{old}]_{new}$ from old to new is given by its inverse, namely \[
\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}.
\]

What are the coordinates of $(4,6)^T$ in this new basis?

One way: Multiply transition matrix by old coordinates $(4,6)^T$ to get new coordinates $(2,-2)^T$. 