Math 310: **Hour Exam 1** (Solutions)
Prof. S. Smith: Fri 17 October 2003

Problem 1: (a) Using Gauss-Jordan elimination, find the row-reduced echelon form of the following augmented matrix: $(A|b) = \left(\begin{array}{rrrr|r} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array}\right)$. Show the STEPS you use.

$$\overset{A_3^{1\times 1}}{\to} \left(\begin{array}{rrrr|r} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{array}\right) \overset{A_3^{4\times 2}}{\to} \left(\begin{array}{rrrr|r} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

(b) Using your answer in (a), give all SOLUTIONS (if any) of the linear equation system $Ax = b$ determined by the augmented matrix $(A|b)$.

Notice columns 2 and 4 have no pivot, so those variables are free.
Then infinite number of solutions: $(5 + 7\alpha - 6\beta, \alpha, -3 + 2\beta, \beta)^T$ *for all real* α, β

Problem 2: (a) Find the LU-decomposition (show method) of the matrix $A = \begin{pmatrix} 6 & 9 \\ 4 & 5 \end{pmatrix}$.

To row-reduce (Gaussian elmination) $\begin{pmatrix} 6 & 9 \\ 4 & 5 \end{pmatrix}$, *we apply* $A_2^{-\frac{2}{3}\times 1}$ *to obtain* $\begin{pmatrix} 6 & 9 \\ 0 & -1 \end{pmatrix}$ *as* U, *and take as* L *the inverse of the matrix for* $A_2^{\frac{2}{3}\times 1}$, *namely* $\begin{pmatrix} 1 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}$

(b) Find the inverse (by any method—but show work) of the matrix $A = \begin{pmatrix} 5 & 10 \\ 4 & 7 \end{pmatrix}$.

By adjoint method, $A^{-1} = -\frac{1}{5} \begin{pmatrix} 7 & -10 \\ -4 & 5 \end{pmatrix}$.

Problem 3: (a) Let S be the subSET of $\mathbf{R}^3$ consisting of all 3-vectors $(x_1, x_2, x_3)^T$ which satisfy the condition $x_1 - 5x_2 - 2x_3 = 0$. Show that S is a subSPACE of $\mathbf{R}^3$.

Vectors in S *have the general form form* $(5a + 2b, a, b)^T$.
(+) Take two general vectors in S: $(5a + 2b, a, b)^T$, $(5c + 2d, c, d)^T$ *and add; their sum is* $(5a + 2b + 5c + 2d, a + c, b + d)^T = (5(a + c) + 2(b + d), a + c, b + d)^T$, *which is also in* S.
(sc.mult.) For a general scalar c, *and general vector* $(5a + 2b, a, b)^T$ *in* S,
the scalar multiple is $((5a + 2b)c, ac, bc)^T = (5(ac) + 2(bc), ac, bc)^T$, *which is also in* S.

(b) Give a basis for the nullspace of the matrix $A = \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{pmatrix}$.
What is the dimension of this space ?

Using $A_1^{-3\times 2}$ *we obtain the row-reduced echelon form of* A $\begin{pmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{pmatrix}$.
The solutions of $Ax = 0$ *are of form* $(7\alpha - 6\beta, -4\alpha + 2\beta, \alpha, \beta)^T$,
so choosing $\alpha = 1, \beta = 0$ *and then* $\alpha = 0, \beta = 1$,
we get a basis given by $(7, -4, 1, 0)^T$ *and* $(-6, 2, 0, 1)^T$.
Therefore the dimension of the nullspace is 2.

Problem 4: (a) Recall that $\mathcal{P}_3$ is the space of polynomials of degree less than three (that is, quadratic polynomials). Are the three "vectors" $1+x+x^2$, $2-2x+2x^2$, and $4+4x^2$ a spanning set for this space ? (Why/why not ?)

Set a linear combination equal to a general vector of the space:

$a(1+x+x^2)+b(2-2x+2x^2)+c(4+4x^2)=d+ex+fx^2$.

Get an equation for each power of x:

(1:) $a+2b+4c=d$

(x:) $a-2b=e$

(x^2:) $a+2b+4c=f$

To the augmented matrix $\left(\begin{array}{rrr|l} 1 & 2 & 4 & d \\ 1 & -2 & 0 & e \\ 1 & 2 & 4 & f \end{array}\right)$ *we apply* $A_3^{-1\times 1}$ *to get* $\left(\begin{array}{rrr|r} 1 & 2 & 4 & d \\ 1 & -2 & 0 & e \\ 0 & 0 & 0 & f-d \end{array}\right)$.

From the third row, we see that ONLY polynomials with $d=f$ (that is, of the form $d+ex+dx^2$) are in the span our our set of 3 vectors; so not all vectors of $\mathcal{P}_3$ are in their span: that is, they are NOT a spanning set for $\mathcal{P}_3$.

(b) Find the coordinates of the vector $(3,2)^T$ in the basis of $\mathbf{R}^2$ given by $(1,1)^T$ and $(2,-1)^T$.

Solve $\left(\begin{array}{rr|r} 1 & 2 & 3 \\ 1 & -1 & 2 \end{array}\right)$ *to get* $(\frac{7}{3},\frac{1}{3})^T$.

Problem 5: (a) If A is a 3×5 matrix of rank 2, what is the nullity (dimension of the nullspace) of A ?

rank + nullity = 5, so nullity = 3.

(b) Use the method of the (classical) adjoint to find the inverse of the matrix $A=\begin{pmatrix} 1 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

First compute $\det(A)=-9$; *so we get* $A^{-1}=-\frac{1}{9}\begin{pmatrix} -9 & -6 & 14 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{pmatrix}$.