Math 310: **Hour Exam 1** (Solutions)
Prof. S. Smith: Mon 27 Sept 1999

**Problem 1:** (a) Using Gauss-Jordan, find the row-reduced echelon form of the following augmented matrix: $(A|b) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & -2 \\ 2 & 6 & 4 & -2 \end{array}\right)$.

$$\stackrel{A_3^{-2\times 1}}{\to} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & 2 & -2 \\ 0 & 2 & 2 & -2 \end{array}\right) \stackrel{M_{\frac{1}{2}\times 1}}{\to} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 2 & -2 \end{array}\right) \stackrel{A_1^{-2\times 2}, A_3^{-2\times 2}}{\to} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(b) Then give the solutions of the corresponding linear system $Ax = b$.
*So solutions are* $(2 + r, -1 - r, r)$ *for free variable* $x_3 = r$.

**Problem 2:** (a) Using Gaussian elimination (best to use only "add" operations), determine an upper-triangular form $U$ (row-echelon, not reduced) for $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 2 \\ 3 & 7 & 10 \end{pmatrix}$.

$$U = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 8 \end{pmatrix}$$

(b) Indicate the ROW OPERATIONS you used to transform $A$ to $U$.
$A_2^{-2\times 1}$, $A_3^{-3\times 1}$, $A_3^{-2\times 2}$ *corrected last, 3/2/05*

(c) What ELEMENTARY ROW MATRICES $E_i$ will, by left muliplication, perform the same operations? (that is, $E_3E_2E_1A = U$)

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}, E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

(d) Give the DETERMINANT of each matrix in (c), and $\det(U)$. How is $\det(U)$ related to $\det(A)$?
$\det(U) = 16$, *and each* $\det(E_i) = 1$; *so as* $E_3E_2E_1A = U$, 1.1.1. $\det(A) = \det(U) = 16$.

**Problem 3:** (a) Find the inverse (any method) of: $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

*Using "adjoint" method:* $\det(A) = -1$, *cofactor matrix is* $\begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix}$; *so inverse* $\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$.

(b) Suppose a country has two political parties $D$ and $R$, and that the "Markov" matrix $A = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ gives the probability of transition among the parties in one year; that is, if $v_0 = (D_0, R_0)^T$ is the original distribution, then one year later the distribution is $Av_0$.
If the original distribution is $v_0 = (\frac{1}{2}, \frac{1}{2})^T$, what is the distribution after 2 years?
*Either compute* $A(Av_0)$; *or* $A^2v_0$ *with* $A^2 = \begin{pmatrix} 1 & \frac{3}{4} \\ 0 & \frac{1}{4} \end{pmatrix}$. *So final distribution is* $(\frac{7}{8}, \frac{1}{8})^T$.

**Problem 4:** (a) Find the determinant (show work) of $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$.

*Top row:* $1(2.9 - 3.4) - 1(1.9 - 1.4) + 1(1.3 - 1.2) = 6 - 5 + 1 = 2.$

(b) Use Cramer's rule to solve $\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 3 & 6 \end{array}\right)$.

$\det(A) = 1.3 - 1.1 = 2$, *so* $x_1 = \frac{1}{2}(3.3 - 6.1) = \frac{3}{2}$ *and* $x_2 = \frac{1}{2}(1.6 - 1.3) = \frac{3}{2}$.

**Problem 5:** (a) Determine the nullspace of the matrix $A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 2 \end{pmatrix}$.

*Just solutions of* $Ax = 0$, *namely* $(3r, -r, r)^t$—*that is, multiples of* $(3, -1, 1)^T$.

(b) Let $V$ be the space of $3 \times 3$ matrices, and $W$ the subSET of "skew-symmetric" matrices: These satisfy $A^T = -A$, and so they have general form $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$. Show $W$ is a subSPACE of $V$.

*(addition:) Take* $A$ *and* $B$ *in* $W$, *compute* $A + B$:

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -e & 0 \end{pmatrix} + \begin{pmatrix} 0 & d & e \\ -d & 0 & f \\ -e & -f & 0 \end{pmatrix} = \begin{pmatrix} 0 & a+d & b+e \\ -a-d & 0 & c+f \\ -b-e & -c-f & 0 \end{pmatrix}.$$

*The result* $A + B$ *is also skew-symmetric, so* $A + B$ *also lies in* $W$.

*(scalar multiplication:) For scalar* $r$ *and* $A \in W$, *compute* $rA \in W$:

$$r \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -e & 0 \end{pmatrix} = \begin{pmatrix} 0 & ra & rb \\ -ra & 0 & rc \\ -rb & -rc & 0 \end{pmatrix}.$$

*The result* $rA$ *is also skew-symmetric, so* $rA$ *also lies in* $W$.

*We see "yes",* $W$ *is a subspace of* $V$.