Math 310: **Hour Exam 1** (Solutions)
Prof. S. Smith: Wed 2 March 2005

Problem 1: (a) Using Gauss-Jordan elimination, find the row-reduced echelon form of the following augmented matrix: $(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 2 \end{array}\right)$. Show the STEPS you use.

$$\overset{A_3^{-1\times 1}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right) \overset{A_1^{-1\times 2} A_3^{1\times 2}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

(b) Using your answer in (a), give all SOLUTIONS (if any) of the linear equation system $Ax = b$ determined by the augmented matrix $(A|b)$.

Notice columns 3 has no pivot, so that variable is free.
Then infinite number of solutions: $(2 - \alpha, -\alpha, \alpha)^T$ *for all real* α.

Problem 2: (a) Find the LU-decomposition (show STEPS) of the matrix $A = \begin{pmatrix} 2 & 3 \\ 4 & 11 \end{pmatrix}$.

To row-reduce (Gaussian elimination) we apply $A_2^{-2\times 1}$ *to obtain* $\begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$ *as* U,

and take as L *the inverse of the matrix for* $A_2^{-2\times 1}$, *namely* $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

(b) Use the row-operations method (show STEPS) to find the inverse of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right) \overset{A_2^{-3\times 1}}{\rightarrow} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right) \overset{M_{-1\times 2}}{\rightarrow} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right) \overset{A_1^{-2\times 2}}{\rightarrow} \left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right)$$

so $A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$

Problem 3: *(a) Use Cramer's rule to solve the system with augmented matrix* $(A|b) = \left(\begin{array}{cc|c} 2 & 1 & 4 \\ 3 & 1 & 5 \end{array}\right)$.

det $A = -1$, *so* $x_1 = \det \begin{pmatrix} 4 & 1 \\ 5 & 1 \end{pmatrix} / -1 = 1$ *and* $x_2 = \det \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix} / -1 = 2$.

(b) Let S *be the subSET of* $\mathbf{R}^3$ *consisting of all 3-vectors* $(x_1, x_2, x_3)^T$ *which satisfy the condition* $x_2 = 2x_3$. *Show that* S *is a subSPACE of* $\mathbf{R}^3$.

Vectors in S *have the general form form* $(a, 2b, b)^T$.
(closure, +) Take two general vectors in S: $(a, 2b, b)^T$, $(c, 2d, d)^T$ *and add; their sum is* $(a + c, 2b + 2d, b + d)^T = (a + c, 2(b + d), b + d)^T$, *which is also in* S.
(closure, sc.mult.) For a general scalar c, and general vector $(a, 2b, b)^T$ *in* S, *the scalar multiple is* $(ac, (2b)c, bc)^T = (ac, 2(bc), bc)^T$, *which is also in* S.

Problem 4: *(a) Do the vectors* $v_1 = (1,0,1)^T$, $v_2 = (0,2,0)^T$, $v_3 = (3,-2,3)^T$ *span* $\mathbf{R}^3$ *? (Why/why not?)*

NO: (short) Notice $v_3 = 3v_1 - 2v_2$*; so their span is 2-dimensional, but* $\mathbf{R}^3$ *is 3-dimensional.*

(longer) Put the v_i *as columns of a matrix augmented by a general vector* $(a,b,c)^T$. $\left(\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 0 & 2 & -2 & b \\ 1 & 0 & 3 & c \end{array}\right)$

Row-reduce to $\left(\begin{array}{ccc|c} 1 & 0 & 3 & a \\ 0 & 1 & -1 & \frac{b}{2} \\ 0 & 0 & 0 & c-a \end{array}\right)$ *and see NOT solvable for every* a,b,c *(only for* $c=a$*)*

(b) Recall that $\mathcal{P}_3$ *is the space of polynomials of degree less than three (i.e., quadratic polynomials). Are the three "vectors"* 1, $1+x$, $1+x+x^2$ *linearly independent in this space ? (Why/why not ?)*

YES: Set an unknown linear combination equal to 0:

$a(1) + b(1+x) + c(1+x+x^2) = 0 = 0.1 + 0.x + 0.x^2$.

Get an equation for each power of x*:*

(1:) $a+b+c=0$

(x*:)* $b+c=0$

(x^2*:)* $c=0$

and check that we get only the zero solution $a=b=c=0$.

E.g., $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ *has row-reduced echelon form* $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$.

Problem 5: *(a) Find a basis for the subspace* S *of* $\mathbf{R}^{2\times 2}$ *given by the upper triangular matrices. What is the dimension of* S *? (Say WHY your choice is a basis.)*

A general upper triangular matrix has form $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$.

So choosing a,b,c *successively in the "standard basis" way, we get 3 "vectors" in a basis:*

$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. *So* $\dim S = 3$.

Why basis? (spanning set:) the linear combination of these with coefficients a,b,c *gives the general matrix* A *above.*

(linearly independent:) Setting that linear combination equal to the zero-matrix gives only the zero solution $a=b=c=0$.

(b) Find the coordinates of the vector $(7,4)^T$ *in the basis of* $\mathbf{R}^2$ *given by* $(1,2)^T$ *and* $(3,5)^T$.

Solve $\left(\begin{array}{cc|c} 1 & 3 & 7 \\ 2 & 5 & 4 \end{array}\right)$ *to get* $(-23,10)^T$.