Math 310: **Hour Exam 1** (Solutions)

Prof. S. Smith: Fri 19 Feb 1999

Problem 1:

(a) Using Gauss-Jordan, find the row-reduced echelon form of the following augmented matrix. (INDICATE the row operations you use).

$$(A|b) = \left(\begin{array}{ccc|c} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{array}\right).$$

$$\overset{A_2^{-2\times 1}, A_3^{-3\times 1}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 4 & 7 & 10 \\ 0 & -3 & -6 & -9 \\ 0 & -6 & -12 & -18 \end{array}\right) \overset{M^{-\frac{1}{3}\times 2}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 4 & 7 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & -6 & -12 & -18 \end{array}\right) \overset{A_1^{-4\times 2}, A_3^{6\times 2}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(b) Then give the solutions of the corresponding linear system $Ax = b$.

So solutions are $(-2 + r, 3 - 2r, r)$ *for free variable* $x_3 = r$.

Problem 2: (a) Given matrices

$$A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 4 \\ 0 & -4 \end{pmatrix},$$

find an elementary 2×2 row matrix E such that left multiplication by E will convert A to B: that is, $EA = B$.

row operation $A_2^{-3\times 1}$ *: add −2 times 1st row to 2nd.*

So use $E = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ *: (obtained by doing that operation to the identity matrix).*

(b) Give the LU-decomposition of

$$A = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix};$$

that is, find lower-triangular L and upper-triangular U, so that $A = LU$.

As in (a) $A_2^{-3\times 1}$ *to get* $U = \begin{pmatrix} 2 & 4 \\ 0 & -4 \end{pmatrix}$. *So* L *from inverse operation* $A_2^{+3\times 1}$ *is* $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

Problem 3: (a) Suppose a population is initially evenly distributed between two states, as represented by the column vector $v = (.5, .5)^T$. Assume the distribution vector after one time unit is given by Av, where A is the transition matrix $\begin{pmatrix} 1 & .5 \\ 0 & .5 \end{pmatrix}$. Compute A^2, and give the distribution vector after 2 time units.

$$A^2 = \begin{pmatrix} 1 & .5 \\ 0 & .5 \end{pmatrix}\begin{pmatrix} 1 & .5 \\ 0 & .5 \end{pmatrix} = \begin{pmatrix} 1 & .75 \\ 0 & .25 \end{pmatrix}, \text{ so } A^2v = A(Av) = \begin{pmatrix} 1 & .75 \\ 0 & .25 \end{pmatrix}\begin{pmatrix} .5 \\ .5 \end{pmatrix} = \begin{pmatrix} .875 \\ .125 \end{pmatrix}.$$

(b) Find the determinant of

$$A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Via top row: $1(4.1 - 0.1) - 5(2.1 - 1.(-1)) + 0 = 1(4) - 5(3) = -11.$

Problem 4:

(a) Find the inverse (any method) of:

$$A = \begin{pmatrix} 3 & -1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}.$$

Using classcal-adjoint method:

$\det(A) = 6$ *(triangular) and matrix of cofactors is* $\begin{pmatrix} 2 & 0 & 0 \\ 2 & 6 & 0 \\ -1 & -9 & 3 \end{pmatrix}$ *so* $A^{-1} = \frac{1}{6}\begin{pmatrix} 2 & 2 & -1 \\ 0 & 6 & -9 \\ 0 & 0 & 3 \end{pmatrix}$

(b) Given matrices $A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$, and also $A^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$; solve the matrix equation $Y = AX + B$—that is, find the 2×2 matrix X.

We see $X = A^{-1}(Y - B) = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ 6 & 6 \end{pmatrix}$

Problem 5: (a) Is $(7, 4, -2)$ in the span of $(1, -2, -5)$ and $(2, 5, 6)$?
Either give coefficients in a linear combination, or explain why it is not possible.

(No): Set up augmented matrix $(A|b) = \left(\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -2 \end{array}\right)$. *Row operations* $A_2^{2\times 1}$ *and* $A_3^{5\times 1}$ *give* $\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 33 \end{array}\right)$. *Now* $M_{\frac{1}{9}\times 2}$ *gives* $\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 33 \end{array}\right)$. *And then* $A_3^{-16\times 2}$ *gives* $\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right)$.

So it is not possible to find coefficients in a linear combination, and then the indicated vector is not in the span. (There are other ways of solving this problem...)

(b) Let $\mathcal{P}_4$ be the space of polynomials of degree less than 4, and W the subSET of polynomials of form $a + bx + (a + b)x^2$ (for all real numbers a, b).
Show that W is a subSPACE of $\mathcal{P}_4$.

Shortcut: notice W *is just the span of* $1 + x^2$ *and* $x + x^2$*, and the span of a set of vectors is always a subspace.*

For the full method: add general vectors $a + bx + (a + b)x^2$ *and* $c + dx + (c + d)x^2$ *from* W *to get* $(a + c) + (b + d)x + (a + b + c + d)x^2$. *We see the coefficient for* x^2 *is the sum of those for* 1 *and* x*, so the sum is also in* W.

Similarly a scalar multiple $f(a+bx+(a+b)x^2 = fa+(fb)x+(fa+fb)x^2$. *Again the coefficient for* x^2 *is the sum of those for* 1 *and* x*, so the scalar multiple is also in* W.

We have closure under both operations, so W *is a subspace.*