Math 310: **Hour Exam 2** (Solutions)
Prof. S. Smith: Mon 9 Nov 1998

You must SHOW WORK to receive credit.

Problem 1:

(a) Are the columns of the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & -3 & 2 \end{pmatrix}$ linearly independent? (Why/why not?)

Yes. One way: $\det(A) = 10 \neq 0$.

(b) Give a basis for the nullspace of the 1×3 matrix $A = (1, 1, 1)$.

Already row-reduced, so x_2 and x_3 are free: solutions of form $(-x_2 - x_3, x_2, x_3)$.

Choosing $x_2 = 1, x_3 = 0$ gives $(-1, 1, 0)$; and $x_2 = 0, x_3 = 1$ gives $(-1, 0, 1)$.

Problem 2:

(a) The *trace* of a 2×2 matrix A is the sum of its diagonal entries; that is, $trace(A) = A_{1,1} + A_{2,2}$. Show that trace is a linear transformation (from 2x2-matrix space to numbers).

(for +:) trace($A+B$)=$(A_{1,1} + B_{1,1}) + (A_{2,2} + B_{2,2})$

trace(A)+trace(B)=$(A_{1,1} + A_{2,2})$ + $(B_{1,1} + B_{2,2})$ ...equal.

(for sc.mult.:) trace(cA)=$cA_{1,1} + cA_{2,2}$; while c.trace(A)=$c(A_{1,1} + A_{2,2})$ ...equal.

(b) Give the matrix representing (in the standard basis) the linear transformation $L : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $L(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$.

Apply L to standard basis, put into columns to get $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix}$

Problem 3:

(a) In $\mathbf{R}^3$ with the usual dot product, determine the orthogonal complement $S^\perp$ to the subspace S spanned by $(1, 1, 1)$ and $(1, 2, 3)$.

Put in as rows of A and row reduce with $A_2^{-1 \times 1}$ to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

Then x_3 is free, and solutions have form $(x_3, -2x_3, x_3)$

(b) Determine the orthogonal (vector) projection of the vector $b = (1, 2, 3)$ in the subspace spanned by the two vectors $v_1 = \frac{1}{\sqrt{2}}(1, -1, 0)$ and $v_2 = \frac{1}{\sqrt{6}}(1, 1, -2)$.

These vectors are already an orthonormal set,

so the scalar projections are just given by the dot products $b \cdot v_1$ and $b \cdot v_2$;

then vector $p = \frac{-1}{\sqrt{2}} v_1 + \frac{-3}{\sqrt{6}} v_2 = -\frac{1}{2}(1, -1, 0) - \frac{1}{2}(1, 1, -2) = (-1, 0, 1)$.

Problem 4:
Find the least-squares solution $\hat{x}$, and the projection p into the column space of A and the error, in the inconsistent system $Ax = b$ given by:

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}.$$

Multiply A^T by the augmented matrix $[A|b]$ to get

$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & -3 \\ 2 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 5 & 2 & 7 \\ 2 & 5 & 7 \end{array}\right).$$

Row operation $A_2^{-\frac{2}{5}\times 1}$ *leads to* $\left(\begin{array}{cc|c} 5 & 2 & 7 \\ 0 & \frac{21}{5} & \frac{21}{5} \end{array}\right)$. *This gives* $x_2 = 1$ *and then* $x_1 = 1$.

Multiply by A to get projection $p = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, *so error is* $b - p = \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix}$, *of length* $\sqrt{21}$.

Problem 5:
(a) Let S be the subspace of $\mathbf{R}^3$ spanned by $v_1 = (1, 0, 1)$ and $v_2 = (1, 1, 0)$. Use the Gram-Schmidt process to find an orthonormal basis for S.

First get orthogonal: use $x_1 = (1, 0, 1)$ *and then*
$x_2 = v_2 - [(v_2 \cdot x_1)/(x_1 \cdot x_1)]x_1 = (1, 1, 0) - [1/2](1, 0, 1) = \frac{1}{2}(1, 2, -1)$.
To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{\sqrt{2}}(1, 0, 1)$ *and* $u_2 = \frac{1}{\sqrt{6}}(1, 2, -1)$.

(b) Now find an orthonormal basis for the orthogonal complement $S^{\perp}$, for S in (a).

To $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ *apply* $A_2^{-1\times 1}$ *to get* $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$
So x_3 is free and solutions have form $(-x_3, x_3, x_3)$;
a basis is given by setting $x_3 = 1$ *to get* $(-1, 1, 1)$.
To make orthonormal, divide by length to get $u_3 = \frac{1}{\sqrt{3}}(-1, 1, 1)$.