Math 310: **Hour Exam 2** (Solutions)
Prof. S. Smith: Fri 10 Nov 2000

You must SHOW WORK to receive credit.
Wherever you use a calculator, write "used calculator".

Problem 1:
(a) Give the matrix A representing (in the standard basis) the linear transformation $L : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $L((x_1, x_2)^T) = (3x_1 - x_2, x_1 + 2x_2)^T$.

Apply L to standard basis, put into columns to get $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$

(b) Now give the matrix B for the same L as in part (a), but using the basis $(1, 1)^T$, $(1, 2)^T$.
Either compute directly with respect to this "new" basis; or use change-of-basis matrix from "new" to "old" basis given by $S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, *and mutliply out* $B = S^{-1}AS$:

$$\begin{pmatrix} 1 & -3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Problem 2: (a) The formula $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\, dx$ defines an inner product on the space $\mathcal{P}_{<2}$ of polynomials of degree less than 2. Show that the functions 1 and x are orthogonal in this inner product. Determine the length $||1||$ fo the function 1 in this inner product.

$\langle 1, x \rangle = \int_{-1}^{1} 1.x\, dx = [\frac{x^2}{2}]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0$, *as desired.*

$\langle 1, 1 \rangle = \int_{-1}^{1} 1.1\, dx = [x]_{-1}^{1} = 1 - (-1) = 2$, *so* $||1|| = \sqrt{2}$.

(b) Find a basis for the orthogonal complement to the row space of $A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 1 & 3 \end{pmatrix}$.

By Thm 5.2.1 this is just the nullspace of A. Compute $rref(A)$: $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

Then variables x_3, x_4 are free, so solutions are $(-x_3, -x_4, x_3, x_4)$. Choosing the free variables in the "standard basis" way, a basis is given by the two vectors $(-1, 0, 1, 0)$ and $(0, -1, 0, 1)$.

Problem 3:
(a) Find all solutions of the system $Ax = b$ given by:

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The row-reduced echelon form of the augmented matrix $[A|b]$ is: $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

The second row says $0 = 1$, so there are no *solutions.*

(b) Find the projection p of b into the column space of A.
This is "least squares": Multiply A^T by the augmented matrix $[A|b]$ to get normal equations

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 2 \\ -1 & -2 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 6 & 12 & 6 \\ 12 & 24 & 12 \end{array}\right). \text{ Compute rref: } \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

*There are an infinite number of solutions for $\hat{x}$, of form $(1 - 2x_2, x_2)^T$;
easiest to take $x_2 = 0$, so $\hat{x} = (1, 0)^T$. Then $p = A\hat{x} = (1, 2, -1)^T$.*

Problem 4:

(a) Let S be the subspace of $\mathbf{R}^3$ (let's use row vectors for convenience) spanned by $v_1 = (1,1,0)$ and $v_2 = (0,1,1)$. Use the Gram-Schmidt process to find an orthonormal basis for S.

First get orthogonal: use $q_1 = v_1 = (1,1,0)$ *and then*
$q_2 = v_2 - [(v_2 \cdot q_1)/(q_1 \cdot q_1)]q_1 = (0,1,1) - [1/2](1,1,0) = (-1/2, 1/2, 1)$.
To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{\sqrt{2}}(1,1,0)$ *and* $u_2 = \frac{1}{\sqrt{6}}(-1,1,2)$.

(b) Now give the projection p of $b = (1,1,1)$ in the space S of part (a).

Could do by least squares; or, as in Thm 5.5.7:
$p = (u_1 \cdot b)u_1 + (u_2 \cdot b)u_2 = \frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}}(1,1,0) + \frac{2}{\sqrt{6}}\frac{1}{\sqrt{6}}(-1,1,2) = (1,1,0) + \frac{1}{3}(-1,1,2) = \frac{2}{3}(1,2,1)$.

Problem 5:

(a) The "trace" of a 2×2 matrix A is the sum of the elements on its main diagonal, namely $Tr(A) = A_{1,1} + A_{2,2}$. Show that the trace function Tr is a *linear* transformation on $\mathbf{R}^{2\times 2}$ (the image space is just scalars $\mathbf{R}$).

(+) $Tr(A+B) = (A+B)_{1,1} + (A+B)_{2,2} = (A_{1,1} + B_{1,1}) + (A_{2,2} + B_{2,2})$
$= (A_{1,1} + A_{2,2}) + (B_{1,1} + B_{2,2}) = Tr(A) + Tr(B)$.
(Or, you can write out general 2×2 *matrices in full).*
(sc.mult.) $Tr(cA) = (cA)_{1,1} + (cA)_{2,2} = cA_{1,1} + cA_{2,2} = c(A_{1,1} + A_{2,2}) = cTr(A)$.

(b) Suppose A is a 3×5 matrix, and L is the usual linear transformation from $\mathbf{R}^5$ to $\mathbf{R}^3$ given by left multiplication: $L(x) = Ax$. If A has rank 3, what is the dimension of the kernel of L ? Why ?

The kernel of L is just the nullspace $N(A)$ *of A, so its dimension is the nullity of A.*
Then (by Thm 3.6.4) $\#(cols\ of A) = 5 = rank(A) + nullity(A)$, *so* $nullity(A) = 2$.