Math 310: Hour Exam 2 (Solutions)
Prof. S. Smith: Mon 18 Nov 2002
You must SHOW WORK to receive credit.
Wherever you use a calculator, write “used calculator”.

Problem 1:
(a) (review from chapter 3)
Show that the set $S$ of vectors $(x_1, x_2)^T$ in $\mathbb{R}^2$ satisfying the condition $x_1 + 2x_2 = 0$
forms a subspace of $\mathbb{R}^2$.

$\text{(add)}$ Assume that $(x_1, x_2)^T$ and $(y_1, y_2)^T$ are in $S$. This means $x_1 + 2x_2 = 0$ and $y_1 + 2y_2 = 0$.
Is the sum of these two vectors, namely $(x_1 + y_1, x_2 + y_2)^T$, also in $S$?
Check the condition: $(x_1 + y_1) + 2(x_2 + y_2) = (x_1 + 2x_2) + (y_1 + 2y_2) = 0 + 0 = 0$, so “yes”.

$\text{(sc.mult.)}$ Assume that $(x_1, x_2)^T$ is in $S$. This means that $x_1 + 2x_2 = 0$.
Is the multiple of this vector by any scalar $c$, namely $c(x_1, x_2)^T$, also in $S$?
Check the condition: $(c x_1 + 2c x_1) = c(x_1 + 2x_2) = c 0 = 0$, so “yes”.

$\text{(Comment: could also work instead with the form } (\alpha, -\frac{\alpha}{2})^T \text{ of vectors in } S)\text{.}$

(b) Show that the function $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $L((x_1, x_2)^T) = (x_1 + x_2, x_1 + 3x_2)^T$
is a linear transformation.

$\text{(add)}$ $L((x_1, x_2)^T + (y_1, y_2)^T) = L((x_1 + y_1, x_2 + y_2)^T)$
$= ((x_1 + y_1) + (x_2 + y_2), (x_1 + y_1) + 3(x_2 + y_2))^T$, while
$L((x_1, x_2)^T) + L((y_1, y_2)^T) = (x_1 + x_2, x_1 + 3x_2)^T + (y_1 + y_2, y_1 + 3y_2)^T$
$= ((x_1 + x_2) + (y_1 + y_2), (x_1 + 3x_2) + (y_1 + 3y_2))^T$, same.

$\text{(sc.mult.)}$ $L(c(x_1, x_2)^T) = L((cx_1, cx_2)^T) = (cx_1 + cx_2, cx_1 + 3(cx_2))^T$, while
$cL((x_1, x_2)^T) = c(x_1 + x_2, x_1 + 3x_2)^T = (c(x_1 + x_2), c(x_1 + 3x_2))^T$, same.

Problem 2:
(a) Give the matrix $A$ representing (in the standard basis) the linear transformation $L: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
defined by $L((x_1, x_2)^T) = (x_1 - 3x_2, 2x_1 + 5x_2)^T$.

Apply $L$ to standard basis, put into columns, to get $A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}$

(b) Now give the matrix $B$ for the same $L$ as in part (a), but using the basis $(1, 1)^T$ and $(1, 2)^T$.
Either compute directly with respect to this “new” basis; or use change-of-basis matrix
from “new” to “old” basis given by $S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, and multiply out $B = S^{-1}AS$:

$\begin{pmatrix} -11 & -22 \\ 9 & 17 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
Problem 3:

(a) Find all exact solutions of the system $Ax = b$ given by:

$$
\begin{pmatrix}
1 & 1 \\
2 & 1 \\
1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1 \\
\end{pmatrix}.
$$

The row-reduced echelon form of the augmented matrix $[A|b]$ is:

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}.
$$

The third row says $0 = 1$, so there are no solutions.

(b) For this $A$ and $b$, find: all “least squares solutions” $\hat{x}$; the projection $p$ of $b$ in the column space of $A$; and the residual (that is, error).

Multiply $A^T$ by the augmented matrix $[A|b]$ to get normal equations

$$
\begin{pmatrix}
1 & 2 & 1 \\
1 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\frac{1}{3} & 1\\
\frac{1}{3} & 1\\
1 & 0\\
\end{pmatrix}
= 
\begin{pmatrix}
6 & 3 & 4 \\
3 & 2 & 2 \\
\end{pmatrix}.
$$

Compute rref:

$$
\begin{pmatrix}
1 & 0 & \frac{2}{3} \\
0 & 1 & 0 \\
\end{pmatrix}.
$$

Thus $\hat{x} = (\frac{2}{3}, 0)^T$; so $p = Ax = 
\begin{pmatrix}
1 & 1 \\
2 & 1 \\
1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\frac{2}{3} \\
0 \\
\end{pmatrix}
= \frac{2}{3}(1, 2, 1)^T$;

with residual vector $r(\hat{x}) = b - p = (1, 1, 1)^T - \frac{2}{3}(1, 2, 1)^T = \frac{1}{3}(1, -1, 1)$ (of size $\frac{1}{3}$).

Problem 4:

(a) Find the vector projection of $(3, 4)^T$ in the direction of $(1, 2)^T$.

$$
\frac{(3, 4)^T \cdot (1, 2)^T}{(1, 2)^T \cdot (1, 2)^T} (1, 2)^T = \frac{11}{5} (1, 2)^T.
$$

(b) Find the subspace orthogonal to the vectors $(2, 1, 2)^T$ and $(1, 0, -1)^T$.

Write vectors as rows of $A$, and compute nullspace of $A$:

$$
\begin{pmatrix}
2 & 1 & 2 \\
1 & 0 & -1 \\
\end{pmatrix}
\text{has rref } \begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 4 \\
\end{pmatrix},
$$

so solutions are $\alpha(1, -4, 1)^T$.

Problem 5:

(a) Let $S$ be the subspace of $\mathbb{R}^3$ spanned by $v_1 = (2, 1, 2)^T$ and $v_2 = (1, 1, 1)^T$. Use the Gram-Schmidt process to find an orthonormal basis for $S$.

First get orthogonal: use $q_1 = v_1 = (2, 1, 2)^T$ and then

$$
q_2 = v_2 - [(v_2 \cdot q_1)/(q_1 \cdot q_1)]q_1 = (1, 1, 1) - [5/9](2, 1, 2) = \frac{1}{9}(-1, 4, -1).
$$

To make orthonormal, divide by lengths to get $u_1 = \frac{1}{3}(2, 1, 2)$ and $u_2 = \frac{1}{\sqrt{18}}(-1, 4, -1)$.

(b) Give the QR-factorization of the matrix $A$ with columns given by $v_1$ and $v_2$ from part (a).

Then $Q$ has columns $u_1$ and $u_2$ from (a), so

$$
\begin{pmatrix}
\frac{2}{\sqrt{18}} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} \\
\frac{1}{\sqrt{18}} & \frac{3}{\sqrt{18}} & -\frac{1}{\sqrt{18}} \\
\frac{2}{\sqrt{18}} & \frac{2}{\sqrt{18}} & 0 \\
\end{pmatrix},
$$

so we can get $R$ as $Q^T A$, namely

$$
\begin{pmatrix}
-\frac{2}{\sqrt{18}} & \frac{1}{\sqrt{18}} & \frac{2}{\sqrt{18}} \\
\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & -\frac{2}{\sqrt{18}} \\
\frac{2}{\sqrt{18}} & \frac{2}{\sqrt{18}} & \frac{2}{\sqrt{18}} \\
\end{pmatrix}
\begin{pmatrix}
2 & 1 \\
1 & 1 \\
2 & 1 \\
\end{pmatrix}
= 
\begin{pmatrix}
3 & \frac{5}{2} \\
0 & \frac{3}{2} \\
\end{pmatrix}.