Math 310: **Hour Exam 2** (Solutions)
Prof. S. Smith: Mon 18 Nov 2002

You must SHOW WORK to receive credit.

Wherever you use a calculator, write "used calculator".

Problem 1:

(a) (review from chapter 3:)
Show that the set S of vectors $(x_1, x_2)^T$ in $\mathbf{R}^2$ satisfying the condition $x_1 + 2x_2 = 0$ forms a subspace of $\mathbf{R}^2$.

(add) Assume that $(x_1, x_2)^T$ and $(y_1, y_2)^T$ are in S. This means $x_1 + 2x_2 = 0$ and $y_1 + 2y_2 = 0$.
Is the sum of these two vectors, namely $(x_1 + y_1, x_2 + y_2)^T$, also in S?
Check the condition: $(x_1 + y_1) + 2(x_2 + y_2) = (x_1 + 2x_2) + (y_1 + 2y_2) = 0 + 0 = 0$, so "yes".
(sc.mult.) Assume that $(x_1, x_2)^T$ is in S. This means that $x_1 + 2x_2 = 0$.
Is the multiple of this vector by any scalar c, namely $c(x_1, x_2)^T$, also in S?
Check the condition: $(c\ x_1 + 2\ c\ x_1) = c(x_1 + 2x_2) = c\ 0 = 0$, so "yes".
(Comment: could also work instead with the form $(\alpha, -\frac{\alpha}{2})^T$ of vectors in S).

(b) Show that the function $L : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $L((x_1, x_2)^T) = (x_1 + x_2, x_1 + 3x_2)^T$ is a linear transformation.

(add) $L((x_1, x_2)^T + (y_1, y_2)^T) = L((x_1 + y_1, x_2 + y_2)^T)$
$= ((x_1 + y_1) + (x_2 + y_2), (x_1 + y_1) + 3(x_2 + y_2))^T$, *while*
$L((x_1, x_2)^T) + L((y_1, y_2)^T) = (x_1 + x_2, x_1 + 3x_2)^T + (y_1 + y_2, y_1 + 3y_2)^T$
$= ((x_1 + x_2) + (y_1 + y_2), (x_1 + 3x_2) + (y_1 + 3y_2))^T$, *same.*
(sc.mult.) $L(c(x_1, x_2)^T) = L((cx_1, cx_2)^T) = (cx_1 + cx_2, cx_1 + 3(cx_2))^T$, *while*
$cL((x_1, x_2)^T) = c(x_1 + x_2, x_1 + 3x_2)^T = (c(x_1 + x_2), c(x_1 + 3x_2))^T$, *same.*

Problem 2:

(a) Give the matrix A representing (in the standard basis) the linear transformation $L : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $L((x_1, x_2)^T) = (x_1 - 3x_2, 2x_1 + 5x_2)^T$.

Apply L to standard basis, put into columns, to get $A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}$

(b) Now give the matrix B for the same L as in part (a), but using the basis $(1, 1)^T$ and $(1, 2)^T$.

Either compute directly with respect to this "new" basis; or use change-of-basis matrix from "new" to "old" basis given by $S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, *and mutliply out $B = S^{-1}AS$:*

$$\begin{pmatrix} -11 & -22 \\ 9 & 17 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Problem 3:

(a) Find all exact solutions of the system $Ax = b$ given by: $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

The row-reduced echelon form of the augmented matrix $[A|b]$ is: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The third row says $0 = 1$, so there are no *solutions.*

(b) For this A and b, find: all "least squares solutions" $\hat{x}$; the projection p of b in the column space of A; and the residual (that is, error).

Multiply A^T by the augmented matrix $[A|b]$ to get normal equations

$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 6 & 3 & 4 \\ 3 & 2 & 2 \end{array}\right)$. *Compute rref:* $\left(\begin{array}{cc|c} 1 & 0 & \frac{2}{3} \\ 0 & 1 & 0 \end{array}\right)$.

Thus $\hat{x} = (\frac{2}{3}, 0)^T$; *so* $p = A\hat{x} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} = \frac{2}{3}(1,2,1)^T$;

with residual vector $r(\hat{x}) = b - p = (1,1,1)^T - \frac{2}{3}(1,2,1)^T = \frac{1}{3}(1,-1,1)$ *(of size* $\frac{1}{\sqrt{3}}$*).*

Problem 4:

(a) Find the vector projection of $(3,4)^T$ in the direction of $(1,2)^T$.

$\frac{(3,4)^T \cdot (1,2)^T)}{(1,2)^T \cdot (1,2)^T)}(1,2)^T = \frac{11}{5}(1,2)^T$.

(b) Find the subspace orthogonal to the vectors $(2,1,2)^T$ and $(1,0,-1)^T$.

Write vectors as rows of A, and compute nullspace of A:

$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$ *has rref* $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \end{pmatrix}$, *so solutions are* $\alpha(1,-4,1)^T$.

Problem 5:

(a) Let S be the subspace of $\mathbf{R}^3$ spanned by $v_1 = (2,1,2)^T$ and $v_2 = (1,1,1)^T$. Use the Gram-Schmidt process to find an orthonormal basis for S.

First get orthogonal: use $q_1 = v_1 = (2,1,2)^T$ *and then*

$q_2 = v_2 - [(v_2 \cdot q_1)/(q_1 \cdot q_1)]q_1 = (1,1,1) - [5/9](2,1,2) = \frac{1}{9}(-1,4,-1)$.

To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{3}(2,1,2)$ *and* $u_2 = \frac{1}{\sqrt{18}}(-1,4,-1)$.

(b) Give the QR-factorization of the matrix A with columns given by v_1 and v_2 from part (a).

Then Q has columns u_1 and u_2 from (a), so $\begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{18}} \\ \frac{1}{3} & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & -\frac{1}{\sqrt{18}} \end{pmatrix}$,

so we can get R as Q^TA, namely $\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & -\frac{1}{\sqrt{18}} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & \frac{5}{3} \\ 0 & \frac{2}{\sqrt{18}} \end{pmatrix}$.