Math 310: **Hour Exam 2** (Solutions)
Prof. S. Smith: Fri 14 Nov 2003

You must SHOW WORK to receive credit.
WHEREVER you use a calculator, write "used calculator".

Problem 1:
(a) Show why each of the functions $L, M : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $L\Big((x_1, x_2)^T\Big) = (x_2, x_1 + x_2)^T$ and $M\Big((x_1, x_2)^T\Big) = (x_2, x_1 + 1)^T$ is–or is NOT—a linear transformation.

L IS linear: (add) $L\Big((x_1, x_2)^T + (y_1, y_2)^T\Big) = L\Big((x_1 + y_1, x_2 + y_2)^T\Big)$
$= \Big(x_2 + y_2, (x_1 + y_1) + (x_2 + y_2)\Big)^T$,
while $L\Big((x_1, x_2)^T\Big) + L\Big((y_1, y_2)^T\Big) = (x_2, x_1 + x_2)^T + (y_2, y_1 + y_2)^T$
$= \Big(x_2 + y_2, (x_1 + x_2) + (y_1 + y_2)\Big)^T$*—same, using commutativity of addition.*
(sc.mult.) $L\Big(c(x_1, x_2)^T\Big) = L\Big((cx_1, cx_2)^T\Big) = \Big(cx_2, cx_1 + cx_2\Big)^T$,
while $cL\Big((x_1, x_2)^T\Big) = c\Big(x_2, x_1 + x_2\Big)^T = (cx_2, c(x_1 + x_2))^T$*—same, using distributive law.*
...but M is NOT linear: (e.g., add:) (add) $M\Big((x_1, x_2)^T + (y_1, y_2)^T\Big) = M\Big((x_1 + y_1, x_2 + y_2)^T\Big)$
$= \Big(x_2 + y_2, (x_1 + y_1) + 1\Big)^T$,
while $M\Big((x_1, x_2)^T\Big) + L\Big((y_1, y_2)^T\Big) = (x_2, x_1 + 1)^T + (y_2, y_1 + 1)^T$
$= \Big(x_2 + y_2, x_1 + y_1 + 2\Big)^T$, NOT the same.

(b) Give the matrix representing (in the standard basis) the linear transformation $L : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $L\Big((x_1, x_2, x_3)^T\Big) = (3x_1 + 4x_2 - 3x_3, 5x_1 + 6x_3, 4x_1 + 3x_2 + 2x_3)^T$.

Apply L to that basis: put into columns, to get $A = \begin{pmatrix} 3 & 4 & -3 \\ 5 & 0 & 6 \\ 4 & 3 & 2 \end{pmatrix}$

Problem 2:
(a) Give the matrix representing the linear transformation $L : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $L\Big((x_1, x_2)^T\Big) = (x_2, x_1 - x_2)^T$, WITH RESPECT TO THE BASIS $(1, 1)^T$, $(1, 2)^T$.

("directly":) Apply L to this basis: $L\Big((1, 1)^T\Big) = (1, 0)^T$; $L\Big((1, 2)^T\Big) = (2, -1)^T$.
Get their coordinates in that basis: $(1, 0)^T = 2(1, 1)^T - 1(1, 2)^T$; $(2, -1)^T = 5(1, 1)^T - 3(1, 2)^T$.
Put in columns, to get $B = \begin{pmatrix} 2 & 5 \\ -1 & -3 \end{pmatrix}$.

(or using "shortcut":) matrix in standard basis (as in 1b) is $A = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$;
obtain B as $S^{-1}AS$ using change of basis matrix $[new]_{std}$ given by
$S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, *so* $S^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$...

(b) Using the "usual" inner product in the space $\mathbf{R}^{2\times 2}$ of matrices (namely $\langle A, B\rangle = \sum_{i,j=1}^{2} A_{i,j}B_{i,j}$), find the vector projection of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in the direction of $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

Projection formula: $\frac{\langle A,B\rangle}{\langle B,B\rangle}B = \frac{1\cdot 1+2\cdot 0+3\cdot 1+4\cdot 0}{1\cdot 1+0\cdot 0+1\cdot 1+0\cdot 0}B = \frac{1+3}{1+1}B = 2B = \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}$

Problem 3:

(a) Find the subspace of $\mathbf{R}^3$ orthogonal to the vectors $(1,2,3)^T$ and $(1,4,5)^T$.

Write vectors as rows of A, and compute nullspace of A:

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}$ *has rref* $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, *so solutions are* $\alpha(1,1,-1)^T$.

(b) Find the coordinates of the vector $(1,2)^T$ in the orthonormal basis of $\mathbf{R}^2$ (for the usual dot product) given by $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T$.

Just take dot products with the basis, to get coordinates: $(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}})^T$.

Problem 4:

(a) For the inconsistent system $Ax = b$ given by: $\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, find: all "least squares solutions" $\hat{x}$; the projection p of b in the column space of A; and the residual (error).

Multiply A^T by the augmented matrix $[A|b]$ to get normal equations

$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{array} \right) = \left(\begin{array}{cc|c} 6 & 1 & 0 \\ 1 & 2 & 1 \end{array} \right)$. *Compute rref:* $\left(\begin{array}{cc|c} 1 & 0 & -\frac{1}{11} \\ 0 & 1 & \frac{6}{11} \end{array} \right)$.

Thus $\hat{x} = \frac{1}{11}(-1,6)^T$; *so* $p = A\hat{x} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 1 \end{pmatrix} \frac{1}{11} \begin{pmatrix} -1 \\ 6 \end{pmatrix} = \frac{1}{11}(-1,4,7)^T$;

with residual vector $r(\hat{x}) = b - p = (1,0,1)^T - \frac{1}{11}(-1,4,7)^T = \frac{1}{11}(12,-4,4)^T$ *(of size* $\frac{4}{\sqrt{11}}$*).*

(b) Find the projection of $(3,0,0)^T$ in the subspace of $\mathbf{R}^3$ spanned by $(1,-1,1)^T$ and $(1,2,1)^T$.

Could do least squares; but the two vectors are orthogonal, so sum of vector projections suffices:

$\frac{(3,0,0)^T \cdot (1,-1,1)^T}{(1,-1,1)^T \cdot (1,-1,1)^T}(1,-1,1)^T + \frac{(3,0,0)^T \cdot (1,2,1)^T}{(1,2,1)^T \cdot (1,2,1)^T}(1,2,1)^T = \frac{3}{3}(1,-1,1)^T + \frac{3}{6}(1,2,1)^T = \frac{1}{2}(3,0,3)^T$

Problem 5:

(a) Let S be the subspace of $\mathbf{R}^3$ spanned by $v_1 = (1,1,0)^T$ and $v_2 = (1,2,2)^T$. Use the Gram-Schmidt process to find an orthonormal basis for S; and give an orthonormal basis for $S^\perp$.

First get orthogonal: use $q_1 = v_1 = (1,1,0)^T$ *and then*

$q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1 = (1,2,2)^T - \frac{3}{2}(1,1,0)^T = \frac{1}{2}(-1,1,4)^T$.

To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{\sqrt{2}}(1,1,0)^T$ *and* $u_2 = \frac{1}{\sqrt{18}}(-1,1,4)^T$.

For $S^\perp$, convert (integer part suffices) to rows $\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 4 \end{pmatrix}$ *and compute rref as* $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix}$;

so $S^\perp$ is span of $(2,-2,1)^T$; *divide by length to get orthonormal basis* $\frac{1}{3}(2,-2,1)^T$.

(b) Give the QR-factorization of the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Apply Gram-Schmidt as in (a) to the columns of A to get $Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

and can obtain R as $Q^T A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$.