Math 310: **Hour Exam 2** (Solutions)
Prof. S. Smith: Fri 17 Nov 2006

You must SHOW WORK to receive credit.

WHEREVER you use a calculator, write "used calculator".

Problem 1:

(a) Give the matrix (with respect to the STANDARD basis) for the linear transformation $L : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 + 2x_2 \\ 3x_1 - x_2 \end{pmatrix}$.

Apply L to the standard basis: $L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ *and* $L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$,

to get the columns of the matrix for the transformation $\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}$.

(b) Now give the matrix for the same linear transformation L as in part (a), but with respect to the basis $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$,

Shortcut method: for A the matrix of (a), with change-of-basis matrix $S = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$,

compute $S^{-1}AS = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 15 & 25 \\ -10 & -17 \end{pmatrix}$

(Or, "directly" as in (a)—columns now from COORDINATES of images in new basis...)

Problem 2:

(a) On the space $\mathcal{P}_3$ of polynomials of degree at most 2, show that the transformation defined by $L(p(x)) = x\ p'(x)$ is LINEAR.

(addition) Take general polynomials $p(x)$ and $q(x)$ from $\mathcal{P}_3$.

Apply L AFTER sum: $L(p(x) + q(x)) = x\ (p(x) + q(x))' = x\ p'(x) + x\ q'(x)$*; now compare*

L BEFORE sum: $L(p(x)) + L(q(x)) = x\ p'(x) + x\ q'(x)$*; equal to above, as desired.*

(scalar multiplication) Take general $p(x) \in \mathcal{P}_3$ and scalar c.

Apply L after sc.mult.: $L(c\ p(x)) = x\ (c\ p(x))' = x\ c\ p'(x)$*; now compare*

L before: $c\ L(p(x)) = c\ x\ p'(x)$*; equal to above, as desired.*

(Or: write $p(x)$ in form $a_0 + a_1x + a_2x^2$, and $q(x)$ as $b_0 + b_1x + b_2x^2$, and apply L as above ...)

(b) For the subspace S of $\mathbf{R}^3$ given by the span of the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$,

find the orthogonal complement $S^\perp$.

Write vectors as rows of A, and compute nullspace of A:

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{pmatrix}$ *has rref* $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, *so solutions are the span of* $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

Problem 3:

(a) For the inconsistent system $Ax = b$ with augmented matrix $\left(\begin{array}{rr|r} -1 & -2 & 1 \\ 2 & 4 & 2 \\ 1 & -2 & 3 \end{array}\right)$, find:
all "least squares solutions" $\hat{x}$; the projection p of b in the column space of A; and the size (length) of the "error" $b - p$.

Multiply A^T by the augmented matrix $[A|b]$ to get normal equations

$$\begin{pmatrix} -1 & 2 & 1 \\ -2 & 4 & -2 \end{pmatrix} \left(\begin{array}{rr|r} -1 & -2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{array}\right) = \left(\begin{array}{rr|r} 6 & 8 & 6 \\ 8 & 24 & 0 \end{array}\right). \textit{ Compute rref: } \left(\begin{array}{rr|r} 1 & 0 & \frac{9}{5} \\ 0 & 1 & -\frac{3}{5} \end{array}\right).$$

$$\textit{Thus } \hat{x} = \begin{pmatrix} \frac{9}{5} \\ -\frac{3}{5} \end{pmatrix}; \textit{ so } p = A\hat{x} = \begin{pmatrix} -1 & -2 \\ 2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} \frac{9}{5} \\ -\frac{3}{5} \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} \\ \frac{6}{5} \\ 3 \end{pmatrix};$$

$$\textit{with error } b - p = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} -\frac{3}{5} \\ \frac{6}{5} \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{8}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix} \textit{ of length } \sqrt{\frac{80}{25}} = \frac{4}{\sqrt{5}}.$$

(b) In the space of differentiable functions on $[0, 1]$, with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\, dx$, find the vector projection of x on the function x^2.

The vector projection formula gives $\frac{\langle x, x^2\rangle}{\langle x^2, x^2\rangle} x^2$.
So compute $\langle x, x^2\rangle = \int_0^1 x \cdot x^2\, dx = \int_0^1 x^3\, dx = [\frac{x^4}{4}]_0^1 = \frac{1}{4}$
and $\langle x^2, x^2\rangle = \int_0^1 x^2 \cdot x^2\, dx = \int_0^1 x^4\, dx = [\frac{x^5}{5}]_0^1 = \frac{1}{5}$,
to get $\frac{\frac{1}{4}}{\frac{1}{5}} = \frac{5}{4}$ *times the function* x^2*; that is, the projection is* $\frac{5}{4}x^2$.

Problem 4:

(a) With inner product space given $\mathbf{R}^2$ and the standard dot product,
find the coordinates of the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ in the orthonormal basis given by $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

Just take dot products with the basis, to get coordinates: $\begin{pmatrix} \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

(b) In the space of differentiable functions on $[-1, 1]$, with inner product $\langle f, g\rangle = \int_{-1}^1 f(x)g(x)\, dx$, the functions 1 and x are orthogonal (given!). Find the projection of $x^{\frac{1}{3}}$ on the span of 1 and x.

Since 1 *and* x *are orthogonal, the projection formula is* $\frac{\langle x^{\frac{1}{3}}, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle x^{\frac{1}{3}}, x\rangle}{\langle x, x\rangle} x$. *So compute*
$\langle 1, 1\rangle = \int_{-1}^1 1 \cdot 1\, dx = \int_{-1}^1 1\, dx = [x]_{-1}^1 = 2;$
$\langle x, x\rangle = \int_{-1}^1 x \cdot x\, dx = \int_{-1}^1 x^2\, dx = [\frac{x^3}{3}]_{-1}^1 = \frac{2}{3};$
$\langle x^{\frac{1}{3}}, 1\rangle = \int_{-1}^1 x^{\frac{1}{3}} \cdot 1\, dx = \int_{-1}^1 x^{\frac{1}{3}}\, dx = [\frac{3}{4}x^{\frac{4}{3}}]_{-1}^1 = 0;$
$\langle x^{\frac{1}{3}}, x\rangle = \int_{-1}^1 x^{\frac{1}{3}} \cdot x\, dx = \int_{-1}^1 x^{\frac{4}{3}}\, dx = [\frac{3}{7}x^{\frac{7}{3}}]_{-1}^1 = \frac{6}{7}.$
So the projection is $\frac{0}{2}1 + \frac{\frac{6}{7}}{\frac{2}{3}}x = \frac{9}{7}x$.

Problem 5:
(a) Apply the Gram-Schmidt process (SHOW steps) to find an orthonormal basis for the column space of $A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$. Then give the QR-factorization of A.

For orthogonal, use $q_1 = v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and then* $q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \frac{7}{5} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Thus $Q = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$ *and can obtain* R *as* $Q^T A = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 5 & 7 \\ 0 & 1 \end{pmatrix}$.

(b) In the space of differentiable functions on $[-1, 1]$, with inner product $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$, find an orthogonal (but not necessarily orthonormal) basis for the subspace spanned by 1 and x^2. (That is, you need not divide by lengths at the end, to get unit vectors).

Here we take $q_1 = v_1 = 1$; *and* $q_2 = v_2 - \frac{\langle v_2, q_1 \rangle}{\langle q_1, q_1 \rangle} q_1 = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} 1$.
So we need to compute $\langle x^2, 1 \rangle = \int_{-1}^{1} x^2 \cdot 1\, dx = \int_{-1}^{1} x^2\, dx = [\frac{x^3}{3}]_{-1}^{1} = \frac{2}{3}$;
we had already computed $\langle 1, 1 \rangle = 2$ *in Problem 4b.*
So we get q_2 *given by* $x^2 - \frac{\frac{2}{3}}{2} 1 = x^2 - \frac{1}{3}$, *and this will be orthogonal to* 1.
So 1 *and* $x^2 - \frac{1}{3}$ *are an orthogonal basis for the span of* 1 *and* x^2.