Math 310: **Hour Exam 2** (Solutions)
Prof. S. Smith: Mon 8 Nov 1999

You must SHOW WORK to receive credit.

Problem 1:

(a) Are the columns of the matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix}$ linearly independent? (Why/why not?)

No. One way: $\det(A) = 0$.

(b) Give a basis for the column space of the matrix A from part (a). What is the rank of A ?

Notice by part (a) that the rank is not 3. But any pair of columns is LI, and so would give a basis; or you can find other possible bases. In particular, the rank of A is 2.

Problem 2:

(a) For a 2×2 matrix A, define the *symmetrization* $Sym(A)$ to be $\frac{1}{2}(A + A^T)$, where T denotes transpose. (As the name suggests, the symmetrization is in fact a symmetric matrix). Show that Sym is a linear transformation (from the matrix space $\mathbf{R}^{2\times 2}$ into that same space).

(for +:) $Sym(A+B) = \frac{1}{2}((A+B)+(A+B)^T) = \frac{1}{2}(A+A^T)+\frac{1}{2}(B+B^T) = Sym(A)+Sym(B)$.

(for sc.mult.:) $Sym(cA) = \frac{1}{2}(cA + (cA)^T) = c \cdot \frac{1}{2}(A + A^T) = c \cdot Sym(A)$.

(b) Give the matrix representing (in the standard basis) the linear transformation $L : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $L(x_1, x_2, x_3) = (2x_1 + 3x_2 - x_3, -2x_2 - x_3, 4x_1 + x_2 - 3x_3)$.

Apply L to standard basis, put into columns to get $\begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & -1 \\ 4 & 1 & -3 \end{pmatrix}$

Problem 3:

(a) In $\mathbf{R}^3$ with the usual dot product, determine the vector projection of $(1, 1, 1)$ on $(1, 2, 3)$.

This is $\dfrac{(1,1,1)\cdot(1,2,3)}{(1,2,3)\cdot(1,2,3)}(1,2,3) = \dfrac{6}{14}(1,2,3)$.

(b) Let $\mathcal{P}_1$ denote the space of linear polynomials ($ax + b$ for $a, b \in \mathbf{R}$); with inner product given by $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\,dx$. Let S be the subspace spanned by the constant function 1; determine the orthogonal complement $S^\perp$ to S in $\mathcal{P}_\infty$.

For general $ax + b$, *compute* $0 = \langle 1, ax + b\rangle = \int_0^1 (1)(ax + b)\,dx = [a\frac{x^2}{2} + bx]_0^1 = \frac{a}{2} + b$.

Then $a = -2b$, *so* $S^\perp$ *contains polynomials* $-2bx + b = b(-2x + 1)$ *for all* b.

Problem 4:
Find the least-squares solution $\hat{x}$, and the projection p into the column space of A, and the error $b-p$; in the inconsistent system $Ax=b$ given by:

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

Multiply A^T by the augmented matrix $[A|b]$ to get

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \left(\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 5 \end{array} \right) = \left(\begin{array}{cc|c} 2 & 0 & 4 \\ 0 & 3 & 8 \end{array} \right).$$

Thus gives $x_1 = 2$ and then $x_2 = \frac{8}{3}$.

Multiply this vector by A to get projection $p = \frac{1}{3}\begin{pmatrix} 2 \\ 8 \\ 14 \end{pmatrix}$, so the error is $b - p = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, of length $\sqrt{\frac{2}{3}}$.

Problem 5:
(a) Let S be the subspace of $\mathbf{R}^3$ spanned by $v_1 = (0,1,1)$ and $v_2 = (1,0,2)$. Use the Gram-Schmidt process to find an orthonormal basis for S.

First get orthogonal: use $x_1 = (0,1,1)$ and then
$x_2 = v_2 - [(v_2 \cdot x_1)/(x_1 \cdot x_1)]x_1 = (1,0,2) - [2/2](0,1,1) = (1,-1,1)$.
To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{\sqrt{2}}(0,1,1)$ and $u_2 = \frac{1}{\sqrt{3}}(1,-1,1)$.

(b) Now find an orthonormal basis for the orthogonal complement $S^\perp$, for S in (a).

To find $S^\perp$, use matrix from original rows $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

and apply $E_{1,2}$ to get row-reduced form $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

So x_3 is free and solutions have form $(-2x_3, -x_3, x_3)$;
a basis is given by setting $x_3 = 1$ to get $(-2,-1,1)$.
To make orthonormal, divide by length to get $u_3 = \frac{1}{\sqrt{6}}(-2,-1,1)$.