Math 310: **Hour Exam 2** (Solutions)

Prof. S. Smith: Fri 7 Apr 2000

You must SHOW WORK to receive credit.

Wherever you use a calculator, write "used calculator".

Problem 1:

(a) Find a basis for the kernel of the linear transformation $L : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $L((x_1, x_2, x_3)^T) = (x_1 - 2x_2, x_2 - x_3)^T$.

Equations: $x_1 - 2x_2 = 0 = x_2 - x_3$; *free variable:* x_3; *solutions* $(2x_3, x_3, x_3)^T$; *so basis* $(2, 1, 1)^T$.

(b) Show that the mapping $L(A) = A^T$, where A^T is the transpose of a 2×2 matrix A, is a linear transformation on the space $\mathbf{R}^{2\times 2}$.

(addition) $L(A + B) = (A + B)^T = A^T + B^T = L(A) + L(B)$

(scalar mult.) $L(cA) = (cA)^T = c(A^T) = cL(A)$.

Problem 2:

(a) Give the matrix A representing (in the standard basis) the linear transformation $L : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $L((x_1, x_2)^T) = (2x_1 + x_2, 4x_1 - 5x_2)^T$.

Apply L *to standard basis, put into columns to get* $A = \begin{pmatrix} 2 & 1 \\ 4 & -5 \end{pmatrix}$

(b) Now give the matrix B for the same L as in part (a), but using the basis $(1, 1)^T$, $(1, -1)^T$.

Either compute directly with respect to this "new" basis; or use change-of-basis matrix from "new" to "old" basis given by $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, *and mutliply out* $B = S^{-1}AS$:

$$\begin{pmatrix} 1 & 5 \\ 2 & -4 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Problem 3:

(a) In $\mathbf{R}^3$ with the usual dot product, determine the vector projection of $(1, 2, 1)$ on $(2, 1, 2)$.

This is $\dfrac{(1, 2, 1) \cdot (2, 1, 2)}{(2, 1, 2) \cdot (2, 1, 2)}(2, 1, 2) = \dfrac{6}{9}(2, 1, 2) = (\dfrac{4}{3}, \dfrac{2}{3}, \dfrac{4}{3})$.

(b) Let S be the span of the vectors $(1, 2, 1)$ and $(2, 1, 2)$. Find the orthogonal complement $S^\perp$.

Put vectors in as rows of matrix A, *solve* $Ax = 0$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right) \xrightarrow{A_2^{-2\times 1}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right) \xrightarrow{M_{-\frac{1}{3}\times 1}} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{A_1^{-2\times 2}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

to conclude $S^\perp$ *consists of the vectors* $(a, 0, -a) = a(1, 0, -1)$.

Problem 4:
The system $Ax = b$ given by:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

is inconsistent. Find the least-squares solution $\hat{x}$, and the projection p of b into the column space of A. Also find the error-vector for this approximation, and the size of that error.

Multiply A^T by the augmented matrix $[A|b]$ to get normal equations

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{array} \right) = \left(\begin{array}{cc|c} 6 & 6 & 6 \\ 6 & 9 & 9 \end{array} \right), \text{ and solve to get } \hat{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $p = A\hat{x} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, *so the error is* $b - p = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, *of length* $\sqrt{2}$.

Problem 5:
(a) Let S be the subspace of $\mathbf{R}^3$ spanned by $v_1 = (0, 1, -1)$ and $v_2 = (1, -1, 1)$. Use the Gram-Schmidt process to find an orthonormal basis for S.

First get orthogonal: use $q_1 = v_1 = (0, 1, -1)$ *and then*
$q_2 = v_2 - [(v_2 \cdot q_1)/(q_1 \cdot q_1)]q_1 = (1, -1, 1) - [-2/2](0, 1, -1) = (1, 0, 0)$.
To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{\sqrt{2}}(0, 1, -1)$ *and* $u_2 = (1, 0, 0)$.

(b) Now find an orthonormal basis for the orthogonal complement $S^\perp$, for S in (a).

Find $S^\perp$ as in problem (3b): $\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array} \right) \overset{E_{1,2}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right) \overset{A_1^{1\times 2}}{\rightarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right)$

Thus solutions have form $(0, a, a)$, *with basis* $(0, 1, 1)$. *Divide by length to get* $\frac{1}{\sqrt{2}}(0, 1, 1)$.