Math 310: **Hour Exam 2** (Solutions)
Prof. S. Smith: Fri 8 Apr 2005

You must SHOW WORK to receive credit.
WHEREVER you use a calculator, write "used calculator".

Problem 1:
(a) Give the matrix (with respect to the STANDARD basis) for the linear transformation $L : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $L\big((x_1, x_2)^T\big) = (2x_1 - x_2, x_1 + 3x_2)^T$,

Apply L to the standard basis: $L\big((1,0)^T\big) = (2,1)^T$; $L\big((0,1)^T\big) = (-1,3)^T$.

These are coordinates in the standard basis, so put in columns, to get $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$.

(b) Now give the matrix for the same linear transformation L as in part (a), but with respect to the basis $(1,2)^T$, $(1,3)^T$.

Shortcut method: for A the matrix of (a) with change-of-basis matrix $S = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$,

compute $S^{-1}AS = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -7 & -13 \\ 7 & 12 \end{pmatrix}$

(Or, can compute "directly" as in (a)—but getting coordinates in NEW basis...)

Problem 2:
(a) Show that the transpose gives a linear transformation on the matrix space $\mathbf{R}^{2\times 2}$ (that is, the function L given by $L(A) = A^T$ for a matrix A).

(addition) $L(A+B) = (A+B)^T = A^T + B^T$ *while* $L(A) + L(B) = A^T + B^T$, *so equal.*
(sc.mult.) $L(cA) = (cA)^T = cA^T$ *while* $cL(A) = cA^T$, *so equal.*
(Or: can write out 2×2 matrices in full, and SHOW the transposing...)

(b) For the subspace S of $\mathbf{R}^3$ given by the span of the vectors $(1,1,2)^T$ and $(1,2,4)^T$, find the orthogonal complement $S^\perp$.

Write vectors as rows of A, and compute nullspace of A:
$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$ *has rref* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$, *so solutions are the span of* $(0,-2,1)^T$.

Problem 3:

(a) For the inconsistent system $Ax = b$ with augmented matrix: $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 2 \\ -1 & -2 & 1 \end{array}\right)$, find all "least squares solutions" $\hat{x}$; the projection p of b in the column space of A; and the residual (error).

Multiply A^T by the augmented matrix $[A|b]$ to get normal equations

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 2 \\ -1 & -2 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 6 & 12 & 6 \\ 12 & 24 & 12 \end{array}\right). \text{ Compute rref: } \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right).$$

Thus $\hat{x} = (1-2\alpha, \alpha)^T$, *so* $p = A\hat{x} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1-2\alpha \\ \alpha \end{pmatrix} = (1, 2, -1)^T$;

with residual vector $b - p = (3,2,1)^T - (1,2,-1)^T = (2,0,2)^T$ *(of size* $\sqrt{8}$*).*

(b) In the space of differentiable functions on $[0,1]$, with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$, find the vector projection of x on the constant function 1.

Projection formula is $\frac{\langle x,1\rangle}{\langle 1,1\rangle}1$.

So compute $\langle x, 1\rangle = \int_0^1 x \cdot 1\,dx = [\frac{x^2}{2}]_0^1 = \frac{1}{2}$ *and* $\langle 1,1\rangle = \int_0^1 1\cdot 1\,dx = [x]_0^1 = 1$, *to get* $\frac{1}{2}$ *times the constant function* 1.

Problem 4:

(a) With the standard dot product on $\mathbf{R}^2$, find the coordinates of the vector $(2,1)^T$ in the orthonormal basis given by $(\frac{3}{5}, \frac{4}{5})^T$, $(-\frac{4}{5}, \frac{3}{5})^T$.

Just take dot products with the basis, to get coordinates: $(2,-1)^T$.

(b) Give the matrix P for projection into the subspace S of $\mathbf{R}^3$ spanned by $(2,2,1)^T$ and $(-2,1,2)^T$.

We need UU^T where the columns of U are an orthonormal basis for S.

The two vectors are already orthogonal, and have length 3.

So for A given by those columns, $U = \frac{1}{3}A$, *and* $P = \frac{1}{9}AA^T = \frac{1}{9}\begin{pmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{pmatrix}$

Problem 5:

(a) Let S be the subspace of $\mathbf{R}^3$ spanned by $v_1 = (2,2,1)^T$ and $v_2 = (2,1,0)^T$. Use the Gram-Schmidt process to find an orthonormal basis for S; and give an orthonormal basis for $S^\perp$.

First get orthogonal: use $q_1 = v_1 = (2,2,1)^T$ *and then*

$q_2 = v_2 - \frac{v_2\cdot q_1}{q_1\cdot q_1}q_1 = (2,1,0)^T - \frac{6}{9}(2,2,1)^T = \frac{1}{3}(2,-1,-2)^T$.

To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{3}(2,2,1)^T$ *and* $u_2 = \frac{1}{3}(2,-1,-2)^T$.

For $S^\perp$, it suffices to work with integer vectors;

convert columns to rows $\begin{pmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \end{pmatrix}$ *and compute rref as* $\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix}$;

so $S^\perp$ is the span of $(\frac{1}{2}, -1, 1)^T$; *divide by length to get orthonormal basis* $\frac{1}{3}(1,-2,2)^T$.

(b) Give the QR-factorization of the matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

Apply Gram-Schmidt as in (a) to the columns of A to get $Q = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

and can obtain R as $Q^TA = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 5 & 7 \\ 0 & 1 \end{pmatrix}$.