

Prof. S. Smith: Fri 9 Apr 1999

You must SHOW WORK to receive credit.

Problem 1:(a) Do the vectors $(1, 1, 2)$, $(3, 2, 1)$, and $(5, 3, 0)$ form a basis of \mathbf{R}^3 ? (indicate why/why not)*No. One way: The matrix with these rows has determinant 0, so the vectors are not LI.*(b) Inside the vector space \mathcal{P}_4 (of polynomials of degree less than 4), what is the dimension of the subspace spanned by $1 - x$, $x - x^2$, $1 - 2x + x^2$? (Indicate a basis for this subspace).*The dimension is 2: since $(1 - x) - (x - x^2) = 1 - 2x + x^2$, we see the three polynomials are LD. But the first two $1 - x$, $x - x^2$ are LI, so form a basis for the span of all 3.***Problem 2:**(a) Let V be the space of all differentiable functions on \mathbf{R} ; and define a mapping

$$L(f(x)) = f'(x) - f(x)$$

on V . Show that L is a LINEAR transformation.*(for +:) $L(f + g) = (f + g)' - (f + g) = f' + g' - f - g$ while* *$L(f) + L(g) = (f' - f) + (g' - g) \dots$ equal.**(for scalar mult. :) $L(cf) = (cf)' - cf = cf' - cf$ while* *$cL(f) = c(f' - f) \dots$ equal. So L is linear.*

(b) Give the matrix representing (in the standard basis) the linear transformation

 $L : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ defined by $L(x_1, x_2, x_3) = (x_1 - 7x_2 + 3x_3, 5x_1 - 3x_3, 4x_1 - x_2 + 4x_3)$.*Apply L to standard basis, put into columns to get*

$$\begin{pmatrix} 1 & -7 & 3 \\ 5 & 0 & -3 \\ 4 & -1 & 4 \end{pmatrix}$$
Problem 3:(a) In \mathbf{R}^3 with the usual dot product, determine the orthogonal complement to the row space of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.*Just solve $Ax = 0$: row-reduce with $A_2^{-1 \times 1}$ to $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$.**Then x_3 is free, and solutions have form $(-x_3, -x_3, x_3)^T$* (b) Verify that the vectors $(1, -1, 0)$, $(1, 1, -2)$ and $(1, 1, 1)$ are orthogonal (in the usual dot product) to each other. This guarantees they are LI, and so form a basis of \mathbf{R}^3 . Determine the coordinates of the vector $b = (1, 2, 3)$ in this basis (easier, if you take advantage of the orthogonality).*The dot product for each pair is indeed 0. So dividing them by their lengths gives an orthonormal basis.**So we do not have to solve systems $Ax = b$ as in Section 3.5; as a shortcut we can just find the vector projections as in Section 5.1. For example, the first is*

$$\frac{(1, 2, 3) \cdot (1, -1, 0)}{(1, -1, 0) \cdot (1, -1, 0)}(1, -1, 0) = -\frac{1}{2}(1, -1, 0)$$

So the first coordinate is $-\frac{1}{2}$. Similarly check that the second coordinate is $-\frac{3}{6} = -\frac{1}{2}$ and the third is $\frac{6}{3} = 2$.

Problem 4:

Find the best possible straight line to fit the 3 data points $(0, 1)$, $(1, 2)$, and $(2, 4)$. That is, use the least-squares method to find unknown (m, b) in the inconsistent system $Ax = y$ given by:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}.$$

Multiply A^T by the augmented matrix $[A|b]$ to get

$$\begin{pmatrix} 5 & 3 & 10 \\ 3 & 3 & 7 \end{pmatrix}$$

Row operation $A_2^{-\frac{3}{5} \times 1}$ leads to $\left(\begin{array}{cc|c} 5 & 3 & 10 \\ 0 & \frac{6}{5} & 1 \end{array} \right)$. This gives $b = \frac{5}{6}$ and then $m = \frac{3}{2}$.

Problem 5:

(a) Let S be the subspace of \mathbf{R}^3 spanned by $v_1 = (2, -1, -1)$ and $v_2 = (-1, 2, -1)$. Use the Gram-Schmidt process to find an orthonormal basis for S .

First get orthogonal: use $x_1 = (2, -1, -1)$ and then

$$x_2 = v_2 - [(v_2 \cdot x_1)/(x_1 \cdot x_1)]x_1 = (-1, 2, -1) - [-3/6](2, -1, -1) = \frac{1}{2}(0, 3, -3).$$

To make orthoNORMAL, divide by lengths to get $u_1 = \frac{1}{\sqrt{6}}(2, -1, -1)$ and $u_2 = \frac{1}{\sqrt{2}}(0, 1, -1)$.

(b) Now find an orthogonal basis for the orthogonal complement S^\perp , for S in (a). (That is, extend your answer in (a) to an orthonormal basis of \mathbf{R}^3).

Working as in Problem 3(a) we find S^* is spanned by $(1, 1, 1)$, so an orthonormal basis is given by $\frac{1}{\sqrt{3}}(1, 1, 1)$.