Math 310: **Final Exam** (Solutions)
Prof. S. Smith: Wed 10 Dec 1997
You must SHOW WORK to receive credit.

Problem 1: Let $A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}$.

(a) Find the eigenvalues of A. (Hint: they are small, positive integers, including 1).
$\det(A - xI) = -x(x^2 - 5x + 6 - 2) + 1(6 - 2x - 2) - 1(4 - (4 - 2x)) = -x^3 + 5x^2 - 8x + 4 = -(x-1)(x^2 - 4x + 4) = -(x-1)(x-2)^2$, *so values* $1, 2, 2$.

(b) Find the eigenspaces for those eigenvalues.

For 1: $A - I = \begin{pmatrix} 2 & 1 & -1 \\ 2 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix}$ *via* $A_{2,3}^{-1\times 1}$ $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ *so solutions* $a(\frac{1}{2}, 0, 1)^T$.

For 2: $A - 2I = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & 2 & -2 \end{pmatrix}$ *via* $A_{2,3}^{-2\times 1}$ $\begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *so solutions* $b(\frac{1}{2}, \frac{1}{2}, 1)^T$.

(c) Is A diagonalizable? Why/why not? If so, find X with $X^{-1}AX$ diagonal.
No: 2 is an eigenvalue twice, but only one dimension of 2-eigenvectors.

Problem 2: Consider the differential equation system (functions of t): $\begin{pmatrix} y_1' & = & & 2y_2 \\ y_2' & = & 3y_1 & -y_2 \end{pmatrix}$.

I GIVE you that $X^{-1}AX = D$ where $A = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$.

(a) Give the *general* solution of the system (with undetermined constants c_1, c_2).
$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{-3t}$ *so* $y_1 = c_1 e^{2t} + 2c_2 e^{-3t}$ *and* $y_2 = c_1 e^{2t} - 3c_2 e^{-3t}$.

(b) Now determine the particular solution (values of c_1, c_2) for the initial value problem $y_1(0) = 5$, $y_2(0) = -5$.
Solve $\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & -3 & -5 \end{array}\right)$ *to get* $c_1 = 1, c_2 = 2$.

Problem 3: Let A be the Markov matrix $\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A. Indicate the "steady-state" vector.
$\det(A - xI) = x^2 - \frac{5}{6}x - \frac{1}{6} + \frac{2}{6} = x^2 - \frac{5}{6}x + \frac{1}{6} = (x-1)(x+\frac{1}{6})$ *so values* $1, -\frac{1}{6}$.
Corresponding eigenvectors $a(4,3)^T$ *and* $b(-1,1)^T$. *Steady-state is* $(\frac{4}{7}, \frac{3}{7})^T$.

(b) Write $A = XDX^{-1}$ with D diagonal—and then get a formula for the power A^m.
We can use $X = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$ *so* $X^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}$

so $A^m = XD^mX^{-1} = \frac{1}{7}\begin{pmatrix} 4 + 3(-\frac{1}{6})^m & 4 - 4(-\frac{1}{6})^m \\ 3 - 3(-\frac{1}{6})^m & 3 + 4(-\frac{1}{6})^m \end{pmatrix}$

Problem 4: Let A be the symmetric matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

I GIVE you that the eigenvalues of A are $2, -1, -1$.

(a) Find an orthonormal basis for each eigenspace.

For 2: eigenvectors are $a(1,1,1)^T$, *so use* $\frac{1}{\sqrt{3}}(1,1,1)^T$.

For -1: Get eigenvectors $(-b-c,b,c)^T$. *Start with basis like* $(-1,1,0)^T$ *and* $(-1,0,1)^T$.

Apply Gram-Schmidt to get $\frac{1}{\sqrt{2}}(-1,1,0)^T$ *and* $\frac{1}{\sqrt{6}}(1,1,-2)^T$.

(b) Now find an orthogonal matrix X (that is, $X^{-1}=X^T$) with $X^{-1}AX$ diagonal.

From (a) can use $X = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix}$.

Problem 5:

(a) Give the (symmetric) matrix for the quadratic form $x_1^2 - 2x_1x_3 + 2x_2^2 + 4x_2x_3 + 6x_3^2$. Indicate new variables in which the form is diagonal. Is it positive definite?

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ -1 & 2 & 6 \end{pmatrix}.$$

Completing squares, we see $(x_1-x_3)^2 + 2(x_2+x_3)^2 + 3x_3^2$.

These give new diagonalizing variables; all with positive coefficient, so positive definite.

(b) Write the matrix of (a) in the form LDL^T with L lower triangular, and D diagonal.

Operations $A_3^{1\times 1}$ *and* $A_3^{-1\times 2}$ *give* $U = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

so $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ *and* $L^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ *and get* L *as* $(L^T)^T$.