Math 310 (2 pm MWF): **Final Exam** (Solutions)
Prof. S. Smith (originally Prof. H. Alexander): Tues 4 May 1999

There are 6 problems, worth 20 % each (so you can get more than 100 %).

You must SHOW WORK to get credit; if by calculator, show WHERE and HOW you used it.

Problem 1: (a) (as promised, a flashback to the midterm):

Let S be the set of all lower triangular 2×2 matrices. Remember that these matrices have the general form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$. Show that S is a subspace of the vector space $\mathbf{R}^{2\times 2}$ of all 2×2 matrices.

(closure, +:) $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} a+d & 0 \\ b+e & c+f \end{pmatrix}$, *also lower-triangular, so in S.*

(closure, scalar mult:) $f \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} fa & 0 \\ fb & fc \end{pmatrix}$, *also lower-triangular, so in S.*

(b) Give the standard matrix representation for linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^2$ defined by $L[(x_1, x_2, x_3)^T] = (x_1 - 3x_2 + 2x_3, 7x_2 + 3x_3)^T$. (That is, just use the standard basis for each space).

Apply L to each basis vector, and write as columns: $\begin{pmatrix} 1 & -3 & 2 \\ 0 & 7 & 3 \end{pmatrix}$

Problem 2: (a) Find the orthogonal complement $S^{\perp}$ to the space spanned by the two vectors $(1,1,1)^T$ and $(1,0,2)^T$.

Write as rows of A: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$. *Then $Ax = 0$ has solutions $x_3(-2,1,1)^T$.*

(b) Find the least-squares solution $\hat{x}$ of the inconsistent system $Ax = b$ given by: $\left(\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 4 \end{array}\right)$.

Give the projection p of b in the column space of A. What is the error?

Multiply by $A^T = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ *to get normal equations* $\left(\begin{array}{cc|c} 5 & 3 & 10 \\ 3 & 3 & 7 \end{array}\right)$.

Solve to get $\hat{x} = \frac{1}{6}(9,5)^T$. Then $p = A\hat{x} = \frac{1}{6}(5,14,23)^T$. And error $= b - p = \frac{1}{6}(1,-2,1)^T$.

Problem 3: (a) Use the Gram-Schmidt process to find an orthonomal basis for the space spanned by the two vectors $b_1 = (0,1,2)^T$ and $b_2 = (1,1,1)^T$.

(Working with rows) $x_1 = (0,1,2)$ and $x_2 = b_2 - \frac{x_2 \cdot x_1}{x_1 \cdot x_1} x_1 = (1,1,1) - = \frac{3}{5}(1,1,1) = (1, \frac{2}{5}, -\frac{1}{5})$
so $u_1 = u_1/|u_1| = \frac{1}{\sqrt{5}}(0,1,2)$ and $u_2 = \frac{1}{\sqrt{30}}(5,2,-1)$.

(b) Find the projection p of the vector $b = (1,2,3)^T$ in the 2-dimensional subspace spanned by the vectors $b_1 = (1,-1,0)^T$ and $b_2 = (1,1,1)^T$.

Many ways of solving. Easiest: note b_1 and b_2 are orthogonal.
So p has coordinates $(b \cdot b_1)/(b_1 \cdot b_1) = -\frac{1}{2}$ and similarly for b_2 to get 2.
so $p = -\frac{1}{2}(1,-1,0)^T + 2(1,1,1)^T = \frac{1}{2}(3,5,4)^T$.

Problem 4: For the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$:

(a) Find the eigenvalues of A.

$\det(A - xI) = x^2 - 5x + 4 - 4 = x^2 - 5x = x(x-5)$, *so eigenvalues are* $0, 5$.

(b) For each eigenvalue, determine the corresponding eigenvectors.

($\lambda = 0$) For $A - 0I = A$ solve $Ax = 0$: solutions are multiples of $(-2, 1)^T$.

($\lambda = 5$) solve $(A - 5I)x = 0$ where $A - 5I = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$ solutions are multiples of $(1, 2)^T$.

Problem 5: Given the differential equation system (y as functions of t): $\begin{pmatrix} y_1' & = & y_1 & + & 2y_2 \\ y_2' & = & 2y_1 & + & y_2 \end{pmatrix}$.

I GIVE you the information that eigenvalues are $3, -1$; with eigenvectors are $(1, 1)^T$ and $(1, -1)^T$.

(a) Give the *general* solution of the system (with undetermined constants c_1, c_2).

$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{pmatrix}$, *so $y_1 = c_1 e^{3t} + c_2 e^{-t}$ and $y_2 = c_1 e^{3t} - c_2 e^{-t}$.*

(b) Now determine the values of c_1, c_2 for the initial value problem $y_1(0) = 4$, $y_2(0) = 2$.

Solve $\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & -1 & 2 \end{array}\right)$ to get $c_1 = 3, c_2 = 1$. So $y_1 = 3e^{3t} + e^{-t}$ and $y_2 = 3e^{3t} - e^{-t}$.

Problem 6: Let A be the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. I GIVE you that the eigenvalues of A are $-1, 1$.

(a) Diagonalize A: that is, find the eigenvalues and eigenvectors, and give a matrix X such that $X^{-1}AX$ is a diagonal matrix D.

For 1: eigenvectors are $a(1, 1)^T$, For -1: Get eigenvectors $b(1, -1)^T$. So $X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

(b) Now let A be the symmetric matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

I GIVE you that the eigenvalues of A are $-1, -1, 2$; and that corresponding eigenvectors are $(1, -1, 0)^T$, $(1, 0, -1)^T$, and $(1, 1, 1)^T$. Give an *orthogonal* diagonalization of A:

That is, use Gram-Schmidt to find an orthonormal basis for each eigenspace. Then use that to build an orthogonal matrix X (that is, $X^{-1} = X^T$) with $X^{-1}AX$ diagonal.

Apply Gram-Schmidt to the -1-eigenspace to get $\frac{1}{\sqrt{2}}(1, -1, 0)^T$ and $\frac{1}{\sqrt{6}}(1, 1, -2)^T$.

For the 2-eigenspace, use $\frac{1}{\sqrt{3}}(1, 1, 1)^T$.

So can use $X = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$.