Math 310: Final Exam                      (Solutions)
Prof. S. Smith: Tues 5 Dec 2000

All 5 problems are worth 20 points each. You must SHOW WORK to receive credit.
(If you use a calculator, WRITE “I used calculator” at those places).

Problem 1: Let \( A = \begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix} \).
(a) Find the characteristic polynomial, and the eigenvalues, of \( A \).
\[ \det(A - x I) = (x - 6)(x + 1) + 12 = x^2 - 5x + 6 = (x - 3)(x - 2), \text{ so eigenvalues are } 2, 3. \]
(b) Find the eigenspaces for those eigenvalues.
\[ \text{For } 2: \quad A - 2.I = \begin{pmatrix} 4 & -4 \\ 3 & -3 \end{pmatrix} \text{ has } \text{ref } \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \text{ get solutions } a(1,1)^T. \]
\[ \text{For } 3: \quad A - 3.I = \begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix}, \text{ has } \text{ref } \begin{pmatrix} 1 & 4/3 \\ 0 & 0 \end{pmatrix}, \text{ get solutions } b(\frac{4}{3},1)^T. \]

Problem 2: Given the differential equation system (functions of \( t \)): \( \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_1 & 2y_2 \\ -2y_1 & y_2 \end{pmatrix} \).
I GIVE you the information that eigenvalues of the coefficient matrix \( A \) for this system are \( 1, -3 \).
(a) Find eigenvectors for \( A \); then use them to give the general solution of the system (with undetermined constants \( c_1, c_2 \)).
\[ \text{For } 1, \text{ get } a(1,1)^T; \text{ for } -1, \text{ get } b(-1,1)^T. \]
\[ \text{Then solution vector } c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} \text{ so } y_1 = c_1 e^t - c_2 e^{-3t} \text{ and } y_2 = c_1 e^t + c_2 e^{-3t}. \]
(b) Now find the particular solution (values of \( c_1, c_2 \)) given initial values \( y_1(0) = 3, y_2(0) = 1 \).
\[ \text{Solve } \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ to get } c_1 = 2, c_2 = -1. \]
\[ \text{So } y_1 = 2e^t + e^{-3t} \text{ and } y_2 = 2e^t - e^{-3t}. \]

Problem 3: (a) For \( A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \), I GIVE you that eigenvalues are \( 1, 2 \). Find eigenvectors for these eigenvalues; and give a matrix \( X \) such that \( X^{-1}AX \) is a diagonal matrix \( D \). (Show \( X^{-1} \) and \( D \) also).
\[ \text{For } 1, \text{ get } a(-3,2)^T; \text{ for } -1, \text{ get } b(-2,1)^T. \]
\[ \text{We can use } X = \begin{pmatrix} -3 \\ 2 \end{pmatrix}, X^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \text{ with } D = X^{-1}AX = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}. \]
(b) Let \( A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \).
\[ \text{GIVEN: the eigenvalues of } A \text{ are } 1, 1, 4. \text{ Find the DIMENSIONS of the eigenspaces for these eigenvalues. (It is not necessary to give eigenvectors). Is } A \text{ diagonalizable? Say why/why not.} \]
\[ \text{Check that } \text{rref}(A - 1.I_3) \text{ has 2 free variables, so the dimension of the 1-eigenspace is 2.} \]
\[ \text{Similarly } \text{rref}(A - 4.I_3) \text{ has 1 free variable, so the dimension of the 4-eigenspace is 1.} \]
\[ \text{Then } A \text{ is diagonalizable—since for each eigenvalue, the dimension of the eigenspace is equal to (not less than) the number of times the eigenvalue appears as a root of the characteristic polynomial. (That is, geometric multiplicity = algebraic multiplicity for each).} \]
Problem 4: Let $A$ be the symmetric matrix \[
\begin{pmatrix}
4 & 2 & -2 \\
2 & 1 & -1 \\
-2 & -1 & 1
\end{pmatrix}.
\]
I GIVE the eigenvalues $0, 0, 6$ of $A$; and an eigenvector $(-2, -1, 1)^T$ for eigenvalue 6.
(a) Find a basis of the eigenspace of $A$ for eigenvalue 0.

The row-reduced echelon form of $A - 0.I = \text{has (1, .5, -.5) as its only nonzero row.}$
So eigenvectors are $(-\frac{1}{2}b + \frac{1}{3}c, b, c)^T$; and one possible basis is $(-1, 2, 0)^T$ and $(1, 0, 2)^T$.
(b) Now find an orthonormal basis for the eigenspace in (a). Use it to give an orthogonal diagonalization of $A$; that is, find an orthogonal matrix $X$ (satisfying $X^{-1} = X^T$) with $X^{-1}AX$ is diagonal.

For 6: eigenspace is 1-dimensional; divide original $(-2, -1, 1)$ by its length $\sqrt{5}$: $x_3 = \frac{1}{\sqrt{5}}(-2, -1, 1)^T$.
For 1: Start with above basis like $v_1 = (-1, 2, 0)^T$ and $v_2 = (1, 0, 2)^T$.
Apply Gram-Schmidt: first $q_1 = (-1, 2, 0)$
and then $q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1}q_1 = (1, 0, 2)^T - \frac{1}{3}(1, 2, 0)^T = (\frac{4}{3}, \frac{2}{3}, 2)^T$
so may as well use the more convenient multiple $q_2 = (2, 1, 5)^T$.
Now divide each by length, to get $x_1 = \frac{1}{\sqrt{3}}(-1, 2, 0)^T$ and $x_2 = \frac{1}{\sqrt{9}}(2, 1, 5)^T$.

So can use $X = \begin{pmatrix}
\frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{3} \\
\frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{3} \\
\frac{1}{\sqrt{6}} & 0 & \frac{1}{3}
\end{pmatrix}$.

Problem 5: (a) The Markov matrix $A = \begin{pmatrix}
.2 & .6 \\
.8 & .4
\end{pmatrix}$, has eigenvalues 1, -.4 and corresponding eigenvectors $(3, 4)^T$ and $(-1, 1)^T$. Use the diagonalization of $A$ to find a formula for the $m$-th power $A^m$, in terms of $m$. What does $A^m$ converge to, for large $m$?

For $X = \begin{pmatrix}
3 & -1 \\
4 & 1
\end{pmatrix}$ and $D = \begin{pmatrix}
1 & 0 \\
0 & .4
\end{pmatrix}$ we have $X^{-1}AX = D$, so $A = XDX^{-1}$.

Thus $A^m = XD^mX^{-1} = \begin{pmatrix}
3 & -1 \\
4 & 1
\end{pmatrix}\begin{pmatrix}
1 & 0 \\
0 & (-.4)^m
\end{pmatrix}^{1 \over 7}\begin{pmatrix}
1 & 1 \\
-4 & 3
\end{pmatrix} = {1 \over 7}\begin{pmatrix}
3 + 4(-.4)^m & 3 - 3(-.4)^m \\
4 - 4(-.4)^m & 4 + 3(-.4)^m
\end{pmatrix}$

Thus $A^m$ for large $m$ converges to $\frac{1}{7}\begin{pmatrix}
3 & 3 \\
4 & 4
\end{pmatrix}$.

(b) Is the symmetric matrix $A = \begin{pmatrix}
4 & 2 & 1 \\
2 & 3 & -2 \\
1 & -2 & 5
\end{pmatrix}$ positive definite? Explain why/why not.

Yes. Hard way: find eigenvalues—messy, but all positive. Easy way: The upper left determinants are $4, 4.3 - 2.2 = 8$, $\det(A) = 13$, all positive.