Math 310: **Final Exam** (Solutions)
Prof. S. Smith: Tues 5 Dec 2000

All 5 problems are worth 20 points each. You must SHOW WORK to receive credit.
(If you use a calculator, WRITE "I used calculator" at those places).

Problem 1: Let $A = \begin{pmatrix} 6 & -4 \\ 3 & -1 \end{pmatrix}$.

(a) Find the characteristic polynomial, and the eigenvalues, of A.

$\det(A - xI) = (x-6)(x+1) + 12 = x^2 - 5x + 6 = (x-3)(x-2)$, so eigenvalues are $2, 3$.

(b) Find the eigenspaces for those eigenvalues.

For 2: $A - 2.I = \begin{pmatrix} 4 & -4 \\ 3 & -3 \end{pmatrix}$ has rref $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, get solutions $a(1,1)^T$.

For 9: $A - 3.I = \begin{pmatrix} 3 & -4 \\ 3 & -4 \end{pmatrix}$, has rref $\begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 0 \end{pmatrix}$, get solutions $b(\frac{4}{3}, 1)^T$.

Problem 2: Given the differential equation system (functions of t): $\begin{pmatrix} y_1' & = & -y_1 & +2y_2 \\ y_2' & = & 2y_1 & -y_2 \end{pmatrix}$.

I GIVE you the information that eigenvalues of the coefficient matrix A for this system are $1, -3$,

(a) Find eigenvectors for A; then use them to give the *general* solution of the system (with undetermined constants c_1, c_2).

For 1, get $a(1,1)^T$; for -1, get $b(-1,1)^T$.

Then solution vector $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}$ so $y_1 = c_1 e^t - c_2 e^{-3t}$ and $y_2 = c_1 e^t + c_2 e^{-3t}$.

(b) Now find the particular solution (values of c_1, c_2) given initial values $y_1(0) = 3$, $y_2(0) = 1$.

Solve $\left(\begin{array}{cc|c} 1 & -1 & 3 \\ 1 & 1 & 1 \end{array}\right)$ to get $c_1 = 2, c_2 = -1$.

So $y_1 = 2e^t + e^{-3t}$ and $y_2 = 2e^t - e^{-3t}$.

Problem 3: (a) For $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$, I GIVE you that eigenvalues are $1, 2$. Find eigenvectors for these eigenvalues; and give a matrix X such that $X^{-1}AX$ is a diagonal matrix D. (Show X^{-1} and D also).

For 1, get $a(-3,2)^T$; for -1, get $b(-2,1)^T$.

We can use $X = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$, $X^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix}$ with $D = X^{-1}AX = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

(b) Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

GIVEN: the eigenvalues of A are $1, 1, 4$. Find the DIMENSIONS of the eigenspaces for these eigenvalues. (It is not necessary to give eigenvectors). Is A diagonalizable? Say why/why not.

Check that $rref(A - 1.I_3)$ has 2 free variables, so the dimension of the 1-eigenspace is 2. Similarly $rref(A - 4.I_3)$ has 1 free variable, so the dimension of the 4-eigenspace is 1. Then A is diagonalizable—since for each eigenvalue, the dimension of the eigenspace is equal to (not less than) the number of times the eigenvalue appears as a root of the characteristic polynomial. (That is, geometric multiplicity = algebraic multiplicity for each).

Problem 4: Let A be the symmetric matrix $\begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$.

I GIVE the eigenvalues $0, 0, 6$ of A; and an eigenvector $(-2, -1, 1)^T$ for eigenvalue 6.

(a) Find a basis of the eigenspace of A for eigenvalue 0.

The row-reduced echelon form of $A - 0.I =$ has $(1, .5, -.5)$ as its only nonzero row.

So eigenvectors are $(-\frac{1}{2}b + \frac{1}{2}c, b, c)^T$; and one possible basis is $(-1, 2, 0)^T$ and $(1, 0, 2)^T$.

(b) Now find an *orthonormal* basis for the eigenspace in (a). Use it to give an orthogonal diagonalization of A; that is, find an *orthogonal* matrix X (satisfying $X^{-1} = X^T$) with $X^{-1}AX$ is diagonal.

For 6: eigenspace is 1-dimensional; divide original $(-2, -1, 1)$ by its length $\sqrt{6}$: $x_3 = \frac{1}{\sqrt{6}}(-2, -1, 1)^T$.

For 1: Start with above basis like $v_1 = (-1, 2, 0)^T$ and $v_2 = (1, 0, 2)^T$.

Apply Gram-Schmidt: first $q_1 = (-1, 2, 0)$

and then $q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1 = (1, 0, 2)^T - \frac{-1}{5}(-1, 2, 0)^T = (\frac{4}{5}, \frac{2}{5}, 2)^T$

so may as well use the more convenient multiple $q_2 = (2, 1, 5)^T$.

Now divide each by length, to get $x_1 = \frac{1}{\sqrt{5}}(-1, 2, 0)^T$ and $x_2 = \frac{1}{\sqrt{30}}(2, 1, 5)^T$.

So can use $X = \begin{pmatrix} -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$.

Problem 5: (a) The Markov matrix $A = \begin{pmatrix} .2 & .6 \\ .8 & .4 \end{pmatrix}$, has eigenvalues 1, $-.4$ and corresponding eigenvectors $(3, 4)^T$ and $(-1, 1)^T$. Use the diagonalization of A to find a formula for the m-th power A^m, in terms of m. What does A^m converge to, for large m ?

For $X = \begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix}$ *and* $D = \begin{pmatrix} 1 & 0 \\ 0 & .4 \end{pmatrix}$ *we have $X^{-1}AX = D$, so $A = XDX^{-1}$.*

Thus $A^m = XD^mX^{-1} = \begin{pmatrix} 3 & -1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-.4)^m \end{pmatrix} \frac{1}{7} \begin{pmatrix} 1 & 1 \\ -4 & 3 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 3 + 4(-.4)^m & 3 - 3(-.4)^m \\ 4 - 4(-.4)^m & 4 + 3(-.4)^m \end{pmatrix}$

Thus A^m for large m converges to $\frac{1}{7} \begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix}$.

(b) Is the symmetric matrix $A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & -2 \\ 1 & -2 & 5 \end{pmatrix}$ positive definite? Explain why/why not.

Yes. Hard way: find eigenvalues—messy, but all positive. Easy way: The upper left determinants are 4, $4.3 - 2.2 = 8$, $det(A) = 13$, all positive.