Math 310: **Final Exam** (Solutions) Prof. S. Smith: Monday 8 Dec 2003

Problem 0: (review: linear transformations)

Let L denote the function from $\mathbf{R}^2$ to $\mathbf{R}^2$ defined by $L(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}) = \begin{pmatrix} x_1 - 2x_2 \\ 3x_1 \end{pmatrix}$.

Show that L is a LINEAR transformation, via the following steps:

(1) Write down a general vector (that is, name its coordinates). Answer: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(2) Write down another general vector (name its coordinates): Answer: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

(3) Now add your two vectors from (1) and (2). Answer: $\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

(3+) Now apply L to the sum vector in (3): Answer: $\begin{pmatrix} (x_1 + y_1) - 2(x_2 + y_2) \\ 3(x_1 + y_1) \end{pmatrix}$

(4) Now apply L separately to the two vectors in (1) and (2).

Answers: $\begin{pmatrix} x_1 - 2x_2 \\ 3x_1 \end{pmatrix}$ and $\begin{pmatrix} y_1 - 2y_2 \\ 3y_1 \end{pmatrix}$

(4+) Now add the two vectors in (4). Answer: $\begin{pmatrix} (x_1 - 2x_2) + (y_1 - 2y_2) \\ 3x_1 + 3y_1 \end{pmatrix}$

FINALLY: Are the answers in (3+) and (4+) the same? Answer: *Yes, using distributive laws*

(5) Write down a general vector (you can use (1) again). Answer: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(6) Write down a general scalar (name it): Answer: c

(7) Now form the product of the vector in (5) and the scalar in (6). Answer: $\begin{pmatrix} c\ x_1 \\ c\ x_2 \end{pmatrix}$

(7+) Now apply L to the product vector in (7): Answer: $\begin{pmatrix} (cx_1) - 2(cx_2) \\ 3(cx_1) \end{pmatrix}$

(8) Now apply L to the vector in (5). Answer: $\begin{pmatrix} x_1 - 2x_2 \\ 3x_1 \end{pmatrix}$.

(8+) Now multiply the vector in (8) by the scalar in (6). Answer: $\begin{pmatrix} c\ (x_1 - 2x_2) \\ c\ (3x_1) \end{pmatrix}$

FINALLY: Are the answers in (7+) and (8+) the same? Answer: *Yes, using distributive laws*

Problem 1: Let $A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$.

(a) Find the characteristic polynomial, and the eigenvalues, of A.

$\det(A - xI) = (2-x)(1-x) - 6 = (x^2 - 3x + 2) - 6 = x^2 - 3x - 4 = (x-4)(x+1)$,
so eigenvalues are $4, -1$.

(b) Find the eigenspaces for those eigenvalues.

For 4: $A - 4.I = \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}$ *has rref* $\begin{pmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{pmatrix}$, *get solutions* $a(\frac{3}{2}, 1)^T$.

For -1: $A + 1.I = \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}$, *has rref* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, *get solutions* $b(-1, 1)^T$.

Problem 2: Given the differential equation system (functions of t): $\begin{pmatrix} y_1' & = & -4y_1 & +2y_2 \\ y_2' & = & 2y_1 & -y_2 \end{pmatrix}$.

I GIVE you the information that eigenvalues of the coefficient matrix A for this system are $0, -5$,

(a) Find eigenvectors for these eigenvalues of A; then use them to give the *general* solution of the system (with undetermined constants c_1, c_2).

For 0, *get* $a(\frac{1}{2}, 1)^T$; *for* -5, *get* $b(-2, 1)^T$.

Then solution vector $c_1 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} e^{0t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-5t}$ *so* $y_1 = \frac{1}{2}c_1 - 2c_2e^{-5t}$ *and* $y_2 = c_1 + c_2e^{-5t}$.

(b) Now find the particular solution (values of c_1, c_2) given initial values $y_1(0) = 0$, $y_2(0) = 5$.

Solve $\left(\begin{array}{cc|c} \frac{1}{2} & -2 & 0 \\ 1 & 1 & 5 \end{array}\right)$ *to get* $c_1 = 4, c_2 = 1$. *So* $y_1 = 2 - 2e^{-5t}$ *and* $y_2 = 4 + e^{-5t}$.

Problem 3:

(a) GIVEN: the eigenvalues of $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ are $2, 3$, with eigenvectors $(-1, 1)^T$ and $(-1, 2)^T$.

Find matrices X, X^{-1}, and D such that $X^{-1}AX = D$ with D a diagonal matrix.
Use this to information find a matrix B such that $B^2 = A$ (use square-root symbols, not decimals).

We can use $X = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$, $X^{-1} = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix}$ *with* $D = X^{-1}AX = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Then $E = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix}$ *satisfies* $E^2 = D$ *(indeed we could multiply either row of* E *by* ± 1

so $B = XEX^{-1} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} - \sqrt{3} & \sqrt{2} - \sqrt{3} \\ -2\sqrt{2} + 2\sqrt{3} & -\sqrt{2} + 2\sqrt{3} \end{pmatrix}$

satisfies $B^2 = A$.

(b) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. GIVEN: the eigenvalues of A are $1, 2, 2$.

Find the DIMENSIONS of the eigenspaces for these eigenvalues.
Is A diagonalizable? Say why/why not.

Check that $rref(A - 1.I_3)$ *has 1 free variable, so the dimension of the 1-eigenspace is 1.*
However $rref(A - 2.I_3)$ *has 2 free variables, so the dimension of the 2-eigenspace is 2.*
Then A *is diagonalizable—since for the eigenvalue 2, the dimension of the eigenspace is equal to the number of times the eigenvalue appears as a root of the characteristic polynomial. (That is, geometric multiplicity = algebraic multiplicity for 2, as well as for 1).*

Problem 4: For the symmetric matrix $A = \begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$,

I GIVE you the eigenvalues $5, -1, -1$ of A; and an eigenvector $(1, -1, 1)^T$ for eigenvalue 5.

(a) Find a basis of the eigenspace of A for eigenvalue -1.

The row-reduced echelon form of $A - (-1).I = A + I$ has $(1, -1, 1)$ as its only nonzero row.
So eigenvectors are $(b - c, b, c)^T$; and one possible basis is $(1, 1, 0)^T$ and $(-1, 0, 1)^T$.

(b) Now find an *orthonormal* basis for the eigenspace in (a).
Use it to give an orthogonal diagonalization of A;
that is, find an *orthogonal* matrix X (satisfying $X^{-1} = X^T$) with $X^{-1}AX$ diagonal.

For 5: eigenspace is 1-dimensional; divide original $(1, -1, 1)$ by its length $\sqrt{3}$: $x_3 = \frac{1}{\sqrt{3}}(1, -1, 1)^T$.
For -1: Start with above basis like $v_1 = (1, 1, 0)^T$ and $v_2 = (-1, 0, 1)^T$.
Apply Gram-Schmidt: first $q_1 = (1, 1, 0)$,
and then $q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1 = (-1, 0, 1)^T - \frac{-1}{2}(1, 1, 0)^T = (-\frac{1}{2}, \frac{1}{2}, 1)^T$
so may as well use the more convenient multiple $q_2 = (-1, 1, 2)^T$.
Now divide each by its length, to get $x_1 = \frac{1}{\sqrt{2}}(1, 1, 0)^T$ and $x_2 = \frac{1}{\sqrt{6}}(-1, 1, 2)^T$.

So now putting x_3 first, can use $X = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$.

Problem 5:

(a) Find a steady-state vector (with coordinates adding to 1) for the Markov matrix $A = \begin{pmatrix} .6 & .3 \\ .4 & .7 \end{pmatrix}$.

Need eigenvector for eigenvalue 1, so $A - 1 \cdot I_2$ has rref $\begin{pmatrix} 1 & -\frac{3}{4} \\ 0 & 0 \end{pmatrix}$,
so divide eigenvector $(3, 4)^T$ by coordinate sum 7 to get proportions $(\frac{3}{7}, \frac{4}{7})^T$.

(b) Is the symmetric matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ positive definite?

Give the decomposition $A = LDL^t$ with L lower triangular and D diagonal.

NOT positive definite: eigenvalues -1,3 not all positive
(or, upper-left determinants 1, -3 not all positive)
For decomposition, we start with the LU-decomposition of chapter 1:
we use $A_2^{-2\times 1}$ to get row-reduced form $U = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$, with $L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$; from U,
we take the diagonal entries to get $D = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$; that is, $U = DL^t$ where $L^t = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.