Math 310: Final Exam  

Prof. S. Smith: Tues 7 Dec 1999  

You must SHOW WORK to receive credit. (If you use a calculator, INDICATE those places where you use it).

Problem 0 (review) is worth 10 points, and Problems 1–5 are worth 20 points. So the maximum score possible is 110.

Problem 0: (As promised, a flashback to Hour Exam 2:)

Let $V$ be the space $\mathbb{F}^{2\times 2}$ of $2 \times 2$ matrices. Let $I$ be the usual $2 \times 2$ identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Determine the orthogonal complement, that is, the subspace $I^\perp$ of all $2 \times 2$ matrices orthogonal to $I$. (Recall that the inner product of two matrices is defined by $\langle A, B \rangle = \sum_{i,j=1}^{2} A_{ij}B_{ij}$).

A is in $I^\perp$ if $0 = \langle A, I \rangle = A_{11}.1 + A_{12}.0 + A_{21}.0 + A_{22}.1 = A_{11} + A_{22}$.

Thus $I^\perp$ consists of the matrices of form $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$.

Problem 1: Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. (a) Find the eigenvalues of $A$.

$\det(A - xI) = (x - 1)^2 - 4 = x^2 - 2x - 3 = (x - 3)(x + 1)$, so eigenvalues are 3, -1.

(b) Find the eigenspaces for those eigenvalues.

For 3: $A - 3.I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$, via $A_2^{-1}\times 1$ to $\begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}$, get solutions $a(1,1)^T$.

For -1: $A - (-1).I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$, via $A_2^{-1}\times 1$ to $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$, get solutions $b(-1,1)^T$.

Problem 2: Given the differential equation system (functions of $t$): $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_1 + y_2 \\ -2y_1 + 4y_2 \end{pmatrix}$.

I GIVE you the information that eigenvalues of the coefficient matrix $A$ for this system are 2, 3, with corresponding eigenvectors $(1,1)^T$ and $(1,2)^T$.

(a) Give the general solution of the system (with undetermined constants $c_1, c_2$).

$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$ so $y_1 = c_1 e^{2t} + c_2 e^{3t}$ and $y_2 = c_1 e^{2t} + 2c_2 e^{3t}$.

(b) Now determine the particular solution (values of $c_1, c_2$) for the initial value problem $y_1(0) = 2, y_2(0) = 1$.

Solve $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$ to get $c_1 = 3, c_2 = -1$.

So $y_1 = 3e^{2t} - e^{3t}$ and $y_2 = 3e^{2t} - 2e^{3t}$.

Problem 3: (a) Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

I GIVE you that the eigenvalues of $A$ are 0, 1, 1. Is $A$ diagonalizable? Indicate why/why not.

Yes: we compute that the eigenspace of 0 is $a(-1,0,1)^T$, and the eigenspace for 1 is $(0,b,c)$ of dimension 2. Thus we CAN find a basis of eigenvectors; alternatively, the geometric multiplicity equals the algebraic multiplicity for each eigenvalue.
(b) For \( A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \), I GIVE you that eigenvalues are 3, 1. Diagonalize \( A \): this is, find \( X \) with \( X^{-1}AX \) diagonal. Use this to determine the exponential \( e^A \).

We compute that eigenvectors are \((1, 1)^T\) and \((1, -1)^T\).

So we can use \( X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \), \( X^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \) with \( D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \).

So \( A = XD X^{-1} \) and then \( e^A = X (e^D) X^{-1} \) where \( e^D = \begin{pmatrix} e^3 & 0 \\ 0 & e \end{pmatrix} \).

Multiplying out we get \( e^A = \frac{1}{2} \begin{pmatrix} e^3 + e & e^3 - e \\ e^3 - e & e^3 + e \end{pmatrix} \).

**Problem 4:** Let \( A \) be the symmetric matrix \( \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \).

I GIVE you that the eigenvalues of \( A \) are \(-2, -2, 1\).

(a) Find a basis for each eigenspace of \( A \).

For \( 1 \): eigenvectors are \( a(1, 1, 1)^T \).

For \(-2\): Get eigenvectors \((b - c, b, c)^T\). So one possible basis is \((-1, 1, 0)^T \) and \((-1, 0, 1)^T \).

(b) Give an orthogonal diagonalization of \( A \); that find an orthogonal matrix \( X \) (satisfying \( X^{-1} = X^T \)) with \( X^{-1}AX \) is diagonal.

For \( 1 \): eigenspace is 1-dimensional; divide previous vector by its length: \( \frac{1}{\sqrt{3}} (1, 1, 1)^T \).

For \(-2\): Start with above basis like \((-1, 1, 0)^T \) and \((-1, 0, 1)^T \).

Apply Gram-Schmidt to get \( \frac{1}{\sqrt{2}} (-1, 1, 0)^T \) and \( \frac{1}{\sqrt{6}} (1, 1, -2)^T \). So can use \( X = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \).

**Problem 5:** (a) For the Markov matrix \( A = \begin{pmatrix} .7 & .2 \\ .3 & .8 \end{pmatrix} \), find the “steady-state” vector \( v \). (That is, \( Av = v \), and the coordinates of \( v \) add up to 1.)

The eigenspace for \( 1 \) consists of vectors \( a(2, 3)^T \). So the steady-state vector is \( (A, .6)^T \).

(b) Is the symmetric matrix \( \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \) positive definite? (Why/why not?)

Yes: e.g., the determinants of the principal minors are positive: \( 1, 1, 1 \). (Alternatively, the eigenvalues are positive; but the computer is needed to find them...)