Math 310: Final Exam (Solutions)
Prof. S. Smith: Tues 2 May 2000

All 5 problems are worth 20 points each. You must SHOW WORK to receive credit.
(If you use a calculator, WRITE “I used calculator” at those places).

**Problem 1:** Let \( A = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix} \).

(a) Find the characteristic polynomial, and the eigenvalues, of \( A \).
\[ \det(A - xI) = (x - 3)(x - 8) - 6 = x^2 - 11x + 18 = (x - 9)(x - 2), \] so eigenvalues are 2, 9.
(b) Find the eigenspaces for those eigenvalues.

For 2: \( A - 2I = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \), via \( A_2^{2 \times 1} \) to \( \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \), get solutions \( a(-2,1)^T \).

For 9: \( A - 9I = \begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix} \), via \( A_2^{1 \times 1} \) to \( \begin{pmatrix} -6 & 2 \\ 0 & 0 \end{pmatrix} \), get solutions \( b(1,1)^T \).

**Problem 2:** Given the differential equation system (functions of \( t \)): \( \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2y_1 + 3y_2 \\ -y_1 - 2y_2 \end{pmatrix} \).
I GIVE you the information that eigenvalues of the coefficient matrix \( A \) for this system are 1, -1.
(a) Find eigenvectors for \( A \); then use them to give the general solution of the system (with undetermined constants \( c_1, c_2 \)).

For 1, get \( a(-3,1)^T \); for \(-1 \), get \( b(-1,1)^T \).

Then solution vector \( c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \) so \( y_1 = -3c_1e^t - c_2e^{-t} \) and \( y_2 = c_1e^t + c_2e^{-t} \).

(b) Now find the particular solution (values of \( c_1, c_2 \)) given initial values \( y_1(0) = 3, y_2(0) = 2 \).

Solve \( \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix} \) to get \( c_1 = -2.5, c_2 = 4.5 \).

So \( y_1 = 7.5e^t - 4.5e^{-t} \) and \( y_2 = -2.5e^t + 4.5e^{-t} \).

**Problem 3:** (a) Let \( A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix} \).


No: As \( A \) is triangular, we see the eigenvalues are 4, 4, 5. Solving \((A - 4I)x = 0\), we see the eigenspace for 4 consists of \( a(0,0,1)^T \): dimension only 1, whereas the eigenvalue is repeated twice. (That is, the geometric multiplicity is less than the algebraic multiplicity). So we cannot get a basis of eigenvectors, and \( A \) is not diagonalizable.

(b) For \( A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \), I GIVE you that eigenvalues are 2, -2; with corresponding eigenvectors \((1,1)^T \) and \((1,-1)^T \). Use the diagonalization of \( A \) (namely use the relevant matrix \( X \) with \( X^{-1}AX \) diagonal) to determine the 5-th power \( A^5 \).

We can use \( X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, X^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \) with \( D = X^{-1}AX = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \).

So \( A = XDX^{-1} \) and then \( A^5 = X(D^5)X^{-1} \) where \( D^5 = \begin{pmatrix} 2^5 & 0 \\ 0 & (-2)^5 \end{pmatrix} \).

Multiplying out we get \( A^5 = \begin{pmatrix} 0 & 32 \\ 32 & 0 \end{pmatrix} \).
Problem 4: Let $A$ be the symmetric matrix 
\[
\begin{pmatrix}
5 & -4 & -2 \\
-4 & 2 & 2 \\
-2 & 2 & 2
\end{pmatrix}
\]
and an eigenvector $(-2, 2, 1)^T$ for eigenvalue $10$. 
(a) Show form of $A - 1.1$ has $(1, -1, -0.5)$ as its only nonzero row. 
(b) Give an orthogonal diagonalization of $A$; that is, find an orthogonal matrix $X$ (satisfying $X^{-1} = X^T$) with $X^{-1}AX$ is diagonal. 
For $10$: space is 1-dimensional; divide original $(-2, 2, 1)$ by its length $3$: \( x_3 = \frac{1}{3}(-2, 2, 1)^T \). 
For $1$: Start with above basis like $v_1 = (1, 1, 0)^T$ and $v_2 = (1, 0, 2)^T$. 
Apply Gram-Schmidt: first $q_1 = (1, 1, 0)$ 
and then $q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1 = (1, 0, 2)^T - \frac{1}{2}(1, 1, 0)^T = (1, -1, 2)^T$ 
so may as well use the more convenient multiple $q_2 = (1, -1, 4)^T$.
Now divide each by length, to get $x_1 = \frac{1}{\sqrt{2}}(1, 1, 0)^T$ and $x_2 = \frac{1}{\sqrt{18}}(1, -1, 4)^T$. 
So can use $X = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{2}{3} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{3}{3} \\
0 & \frac{1}{\sqrt{18}} & \frac{1}{3}
\end{pmatrix}$.

Problem 5: (a) For the Markov matrix $A = \begin{pmatrix} .1 & .6 \\ .9 & .4 \end{pmatrix}$, find the eigenvalues; also find the “steady-state” vector $v$. (That is, $Av = v$, and the coordinates of $v$ add up to 1.) 
\[
\det(A - xI) = (1 - x)(.4 - x) - (.9)(.6) = x^2 - .5x + .04 - .54 = x^2 - .5x - .5 = (x - 1)(x + .5)
\]
so the eigenvalues are $1, -0.5$. 
The eigenspace for $1$ consists of vectors $a(2, 3)^T$. So the steady-state vector is $(4, .6)^T$.
(b) Write the matrix $\begin{pmatrix} 9 & -3 \\ -3 & 2 \end{pmatrix}$ as a product $LDL^T$, with $L$ lower triangular, and $D$ diagonal. 
First get $LU$-decomposition as in Section 1.4: 
$A_2^{1 \times 1}$ takes $A$ to $U = \begin{pmatrix} 9 & -3 \\ 0 & 1 \end{pmatrix}$ 
so that $L = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ 
then factor $U = DU^*$ by using diagonal values of $U$ in $D$: 
$D = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$ so that $U^* = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}$, and then observe that $U^*$ is indeed $L^T$. 