Math 310: **Final Exam** (Solutions)
Prof. S. Smith: Tues 2 May 2000

All 5 problems are worth 20 points each. You must SHOW WORK to receive credit.
(If you use a calculator, WRITE "I used calculator" at those places).

Problem 1: Let $A = \begin{pmatrix} 3 & 2 \\ 3 & 8 \end{pmatrix}$.

(a) Find the characteristic polynomial, and the eigenvalues, of A.

$\det(A - xI) = (x-3)(x-8) - 6 = x^2 - 11x + 18 = (x-9)(x-2)$, so eigenvalues are $2, 9$.

(b) Find the eigenspaces for those eigenvalues.

For 2: $A - 2.I = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$, via $A_2^{-2\times 1}$ to $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, get solutions $a(-2,1)^T$.

For 9: $A - 9.I = \begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix}$, via $A_2^{\frac{1}{2}\times 1}$ to $\begin{pmatrix} -6 & 2 \\ 0 & 0 \end{pmatrix}$, get solutions $b(\frac{1}{3},1)^T$.

Problem 2: Given the differential equation system (functions of t): $\begin{pmatrix} y_1' & = & 2y_1 & +3y_2 \\ y_2' & = & -y_1 & -2y_2 \end{pmatrix}$.

I GIVE you the information that eigenvalues of the coefficient matrix A for this system are $1, -1$,

(a) Find eigenvectors for A; then use them to give the *general* solution of the system (with undetermined constants c_1, c_2).

For 1, get $a(-3,1)^T$; for -1, get $b(-1,1)^T$.

Then solution vector $c_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$ so $y_1 = -3c_1e^t - c_2e^{-t}$ and $y_2 = c_1e^t + c_2e^{-t}$.

(b) Now find the particular solution (values of c_1, c_2) given initial values $y_1(0) = 3$, $y_2(0) = 2$.

Solve $\left(\begin{array}{cc|c} -3 & -1 & 3 \\ 1 & 1 & 2 \end{array}\right)$ to get $c_1 = -2.5, c_2 = 4.5$.

So $y_1 = 7.5e^t - 4.5e^{-t}$ and $y_2 = -2.5e^t + 4.5e^{-t}$.

Problem 3: (a) Let $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$.

Is A diagonalizable? Indicate why/why not.

No: As A is triangular, we see the eigenvalues are 4,4,5. Solving $(A - 4I)x = 0$, we see the eigenspace for 4 consists of $a(0,0,1)^T$: dimension only 1, whereas the eigenvalue is repeated twice. (That is, the geometric multiplicity is less than the algebraic multiplicity). So we cannot get a basis of eigenvectors, and A is not diagonalizable.

(b) For $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, I GIVE you that eigenvalues are $2, -2$; with corresponding eigenvectors $(1,1)^T$ and $(1,-1)^T$. Use the diagonalization of A (namely use the relevant matrix X with $X^{-1}AX$ diagonal) to determine the 5-th power A^5.

We can use $X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $X^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ with $D = X^{-1}AX = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$.

So $A = XDX^{-1}$ and then $A^5 = X(D^5)X^{-1}$ where $D^5 = \begin{pmatrix} 2^5 & 0 \\ 0 & (-2)^5 \end{pmatrix}$.

Multiplying out we get $A^5 = \begin{pmatrix} 0 & 32 \\ 32 & 0 \end{pmatrix}$.

Problem 4: Let A be the symmetric matrix $\begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & 2 \\ -2 & 2 & 2 \end{pmatrix}$.

I GIVE the eigenvalues $1, 1, 10$ of A; and an eigenvector $(-2, 2, 1)^T$ for eigenvalue 10.

(a) Find a basis for the eigenspace of A for eigenvalue 1.

The row-reduced echelon form of $A - 1.I$ has $(1, -1, -.5)$ as its only nonzero row.

So eigenvectors are $(b + \frac{1}{2}c, b, c)^T$; and one possible basis is $(1, 1, 0)^T$ and $(\frac{1}{2}, 0, 1)^T$.

(b) Give an orthogonal diagonalization of A; that is, find an *orthogonal* matrix X (satisfying $X^{-1} = X^T$) with $X^{-1}AX$ is diagonal.

For 10: space is 1-dimensional; divide original $(-2, 2, 1)$ by its length 3: $x_3 = \frac{1}{3}(-2, 2, 1)^T$.

For 1: Start with above basis like $v_1 = (1, 1, 0)^T$ and $v_2 = (1, 0, 2)^T$.

Apply Gram-Schmidt: first $q_1 = (1, 1, 0)$

and then $q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1 = (1, 0, 2)^T - \frac{1}{2}(1, 1, 0)^T = (\frac{1}{2}, -\frac{1}{2}, 2)^T$

so may as well use the more convenient multiple $q_2 = (1, -1, 4)^T$.

Now divide each by length, to get $x_1 = \frac{1}{\sqrt{2}}(1, 1, 0)^T$ and $x_2 = \frac{1}{\sqrt{18}}(1, -1, 4)^T$.

So can use $X = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{1}{3} \end{pmatrix}$.

Problem 5: (a) For the Markov matrix $A = \begin{pmatrix} .1 & .6 \\ .9 & .4 \end{pmatrix}$, find the eigenvalues; also find the "steady-state" vector v. (That is, $Av = v$, and the coordinates of v add up to 1.)

$\det(A - xI) = (.1 - x)(.4 - x) - (.9)(.6) = x^2 - .5x + .04 - .54 = x^2 - .5x - .5 = (x - 1)(x + .5)$

so the eigenvalues are $1, -.5$.

The eigenspace for 1 consists of vectors $a(2, 3)^T$. So the steady-state vector is $(.4, .6)^T$.

(b) Write the matrix $\begin{pmatrix} 9 & -3 \\ -3 & 2 \end{pmatrix}$ as a product LDL^T, with L lower triangular, and D diagonal.

First get LU-decomposition as in Section 1.4:

$A_2^{\frac{1}{3}\times 1}$ *takes A to* $U = \begin{pmatrix} 9 & -3 \\ 0 & 1 \end{pmatrix}$

so that $L = \begin{pmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{pmatrix}$

then factor $U = DU^*$ by using diagonal values of U in D:

$D = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$ *so that* $U^* = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{pmatrix}$, *and then observe that U^* is indeed L^T.*