Math 310: Final Exam  
(Solutions)  
Prof. S. Smith: Tuesday 3 May 2005

Problem 1: Let \( A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} \). Work by hand; do not use a calculator on this problem.

(Except possibly to check your work.)

(a) Find the characteristic polynomial, and the eigenvalues, of \( A \).
\[
\det(A - xI) = (3 - x)(1 - x) - 4.2 = (x^2 - 4x + 3) - 8 = x^2 - 4x - 5 = (x - 5)(x + 1),
\]
so eigenvalues are 5, -1.

(b) Find the eigenspaces for these eigenvalues.
- For 5: \( A - 5I = \begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix} \) has rref \( \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \), get solutions \( a(1,1)^T \).
- For -1: \( A - (-1)I = A + I = \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} \), has rref \( \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \), get solutions \( b(-\frac{1}{2},1)^T \).

Problem 2: Given the differential equation system (functions of \( t \)): \[
\begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} -y_1 + 2y_2 \\ 2y_1 - y_2 \end{pmatrix}.
\]
I GIVE you the information that eigenvalues of the coefficient matrix \( A \) for this system are -3,1.

(a) Find eigenvectors for these eigenvalues of \( A \); then use them to give the general solution of the system (with undetermined constants \( c_1, c_2 \)).
- For -5, get \( a(-1,1)^T \); for 1, get \( b(1,1)^T \).

Then solution vector \( c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t \), so \( y_1 = -c_1 e^{-3t} + c_2 e^t \) and \( y_2 = c_1 e^{-3t} + c_2 e^t \).

(b) Now find the particular solution (values of \( c_1, c_2 \)) given initial values \( y_1(0) = 3, y_2(0) = 1 \).

Solve \( \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \) to get \( c_1 = -1, c_2 = 2 \). So \( y_1 = e^{-3t} + 2e^t \) and \( y_2 = -e^{-3t} + 2e^t \).

Problem 3:

(a) GIVEN: the eigenvalues of \( A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix} \) are 2,1. Diagonalize \( A \): that is, give matrices \( X, X^{-1}, \) and \( D \) such that \( X^{-1}AX = D \) with \( D \) a diagonal matrix.

Find eigenvectors for 2, say \((-2,1)^T\); and for 1, say \((-3,2)^T\).

We can use \( X = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}, X^{-1} = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix} \) with \( D = X^{-1}AX = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \).

(b) Let \( A = \begin{pmatrix} -4 & -10 \\ 3 & -7 \\ 0 & 0 \end{pmatrix} \). GIVEN: the eigenvalues of \( A \) are 2,1,1.

Find the dimensions of the eigenspaces for these eigenvalues.
Is \( A \) diagonalizable? Say why/why not.

Check that rref \( (A - 2I_3) \) has 1 free variable, so the dimension of the 2-eigenspace is 1.
However also rref \( (A - 1I_3) \) has 1 free variable, so the dimension of the 1-eigenspace is only 1.
Then \( A \) is not diagonalizable—since for the eigenvalue 1, the dimension of the eigenspace is less than the number of times the eigenvalue appears as a root of the characteristic polynomial.
(That is, geometric multiplicity < algebraic multiplicity for 1).
Problem 4: For the symmetric matrix 

$$A = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix},$$

I GIVE you the eigenvalues 6, 0, 0 of A; and an eigenvector \((-2, -1, 1)^T\) for eigenvalue 6.

(a) Find a basis of the eigenspace of A for eigenvalue 0.

The row-reduced echelon form of \(A - (0)\cdot I = A\) has \((1, \frac{1}{2}, -\frac{1}{2})\) as its only nonzero row.

So eigenvectors are \((-\frac{1}{2}b + \frac{1}{2}c, b, c)^T\); and one possible basis is \((-1, 2, 0)^T\) and \((1, 0, 2)^T\).

(b) Now find an orthonormal basis for the eigenspace in (a).

Use it to give an orthogonal diagonalization of A;

that is, find an orthonormal matrix \(X\) (satisfying \(X^{-1} = X^T\)) with \(X^{-1}AX\) diagonal.

Show WORK in obtaining your orthonormal basis (no calculators!)

For 6: eigenspace is 1-dimensional; divide original \((-2, -1, 1)\) by its length \(\sqrt{6}\): \(x_3 = \frac{1}{\sqrt{6}}(-2, -1, 1)^T\).

For 0: Start with above basis like \(v_1 = (-1, 2, 0)^T\) and \(v_2 = (1, 0, 2)^T\).

Apply Gram-Schmidt: first \(q_1 = (-1, 2, 0)\),

and then \(q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1 = (1, 0, 2)^T - \frac{1}{3}(-1, 2, 0)^T = (\frac{1}{3}, \frac{2}{3}, 2)^T\)

so may as well use the more convenient multiple \(q_2 = (2, 1, 5)^T\).

Now divide each by its length, to get \(x_1 = \frac{1}{\sqrt{6}}(-1, 2, 0)^T\) and \(x_2 = \frac{1}{\sqrt{30}}(2, 1, 5)^T\).

So now putting \(x_3\) first, can use \(X = \begin{pmatrix} -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & 0 \end{pmatrix} \).

(c) Give the projection matrices into the two eigenspaces for A. (Calculators OK on this part.)

Just compute \(UU^T\), where the columns of \(U\) are the orthonormal basis from (b):

for 6, \(\frac{1}{6} \begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}\) and for 0, \(\frac{1}{6} \begin{pmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{pmatrix}\)

Problem 5: (a) Can you find an orthonormal basis of eigenvectors for 

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}?$$

(Why/why not?)

No; the matrix is not normal (does not commute with its transpose).

So the separate eigenspaces are not orthogonal,

(b) For the Markov matrix \(A = \begin{pmatrix} .8 & -.1 \\ .2 & .9 \end{pmatrix}\), I GIVE you that the eigenvalues are 1 and .7.

Give a formula for the \(n\)th power \(A^n\).

Compute eigenvectors for 1, say \((1, 2)^T\); and for .7, say \((-1, 1)^T\).

So for \(X = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\) we have \(X^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\) and \(X^{-1}AX = D\) where \(D = \begin{pmatrix} 1 & 0 \\ 0 & .7 \end{pmatrix}\).

So \(A = XDX^{-1}\) and hence \(A^n = XD^nX^{-1} = \frac{1}{3} \begin{pmatrix} 1 + 2(.7)^n & 1 - (.7)^n \\ 2 - 2(.7)^n & 2 + (.7)^n \end{pmatrix}\).