Math 310: **Final Exam** (Solutions) Prof. S. Smith: Tuesday 3 May 2005

Problem 1: Let $A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$. Work **by hand**; do *not* use a calculator on this problem. (Except possibly to *check* your work.)

(a) Find the characteristic polynomial, and the eigenvalues, of A.

$\det(A - xI) = (3-x)(1-x) - 4.2 = (x^2 - 4x + 3) - 8 = x^2 - 4x - 5 = (x-5)(x+1)$,
so eigenvalues are $5, -1$.

(b) Find the eigenspaces for those eigenvalues.

For 5: $A - 5.I = \begin{pmatrix} -2 & 2 \\ 4 & -4 \end{pmatrix}$ *has rref* $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, *get solutions* $a(1,1)^T$.

For -1: $A - (-1).I = A + I = \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix}$, *has rref* $\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$, *get solutions* $b(-\frac{1}{2}, 1)^T$.

Problem 2: Given the differential equation system (functions of t): $\begin{pmatrix} y_1' & = & -y_1 & +2y_2 \\ y_2' & = & 2y_1 & -y_2 \end{pmatrix}$.

I GIVE you the information that eigenvalues of the coefficient matrix A for this system are $-3, 1$.

(a) Find eigenvectors for these eigenvalues of A; then use them to give the *general* solution of the system (with undetermined constants c_1, c_2).

For -5, *get* $a(-1,1)^T$; *for* 1, *get* $b(1,1)^T$.

Then solution vector $c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t$, *so* $y_1 = -c_1e^{-3t} + c_2e^t$ *and* $y_2 = c_1e^{-3t} + c_2e^t$.

(b) Now find the particular solution (values of c_1, c_2) given initial values $y_1(0) = 3$, $y_2(0) = 1$.

Solve $\left(\begin{array}{cc|c} -1 & 1 & 3 \\ 1 & 1 & 1 \end{array}\right)$ *to get* $c_1 = -1, c_2 = 2$. *So* $y_1 = e^{-3t} + 2e^t$ *and* $y_2 = -e^{-3t} + 2e^t$.

Problem 3:

(a) GIVEN: the eigenvalues of $A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$ are $2, 1$. Diagonalize A: that is, give matrices X, X^{-1}, and D such that $X^{-1}AX = D$ with D a diagonal matrix.

Find eigenvectors for 2, say $(-2,1)^T$; *and for 1, say (-3,2).*

We can use $X = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}$, $X^{-1} = \begin{pmatrix} -2 & -3 \\ 1 & 2 \end{pmatrix}$ *with* $D = X^{-1}AX = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

(b) Let $A = \begin{pmatrix} -4 & -10 & 2 \\ 3 & 7 & -1 \\ 0 & 0 & 1 \end{pmatrix}$. GIVEN: the eigenvalues of A are $2, 1, 1$.

Find the DIMENSIONS of the eigenspaces for these eigenvalues.

Is A diagonalizable? Say why/why not.

Check that $rref(A - 2.I_3)$ *has 1 free variable, so the dimension of the 2-eigenspace is 1.*
However also $rref(A - 1.I_3)$ *has 1 free variable, so the dimension of the 1-eigenspace is only 1.*
Then A is not diagonalizable*—since for the eigenvalue 1, the dimension of the eigenspace is less than the number of times the eigenvalue appears as a root of the characteristic polynomial. (That is, geometric multiplicity < algebraic multiplicity for 1).*

Problem 4: For the symmetric matrix $A = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$,

I GIVE you the eigenvalues $6, 0, 0$ of A; and an eigenvector $(-2, -1, 1)^T$ for eigenvalue 6.

(a) Find a basis of the eigenspace of A for eigenvalue 0.

The row-reduced echelon form of $A - (0).I = A$ has $(1, \frac{1}{2}, -\frac{1}{2})$ as its only nonzero row.

So eigenvectors are $(-\frac{1}{2}b + \frac{1}{2}c, b, c)^T$; and one possible basis is $(-1, 2, 0)^T$ and $(1, 0, 2)^T$.

(b) Now find an *orthonormal* basis for the eigenspace in (a).

Use it to give an orthogonal diagonalization of A;

that is, find an *orthogonal* matrix X (satisfying $X^{-1} = X^T$) with $X^{-1}AX$ diagonal.

Show WORK in obtaining your orthonormal basis (**no** calculators!)

For 6: eigenspace is 1-dimensional; divide original $(-2, -1, 1)$ by its length $\sqrt{6}$: $x_3 = \frac{1}{\sqrt{6}}(-2, -1, 1)^T$.

For 0: Start with above basis like $v_1 = (-1, 2, 0)^T$ and $v_2 = (1, 0, 2)^T$.

Apply Gram-Schmidt: first $q_1 = (-1, 2, 0)$,

and then $q_2 = v_2 - \frac{v_2 \cdot q_1}{q_1 \cdot q_1} q_1 = (1, 0, 2)^T - \frac{-1}{5}(-1, 2, 0)^T = (\frac{4}{5}, \frac{2}{5}, 2)^T$

so may as well use the more convenient multiple $q_2 = (2, 1, 5)^T$.

Now divide each by its length, to get $x_1 = \frac{1}{\sqrt{5}}(-1, 2, 0)^T$ and $x_2 = \frac{1}{\sqrt{30}}(2, 1, 5)^T$.

So now putting x_3 first, can use $X = \begin{pmatrix} -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$.

(c) Give the projection matrices into the two eigenspaces for A. (Calculators OK on this part.)

Just compute UU^T, where the columns of U are the orthonormal basis from (b):

for 6, $\frac{1}{6}\begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$ *and for 0,* $\frac{1}{6}\begin{pmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{pmatrix}$

Problem 5: (a) Can you find an orthonormal basis of eigenvectors for $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$?

(Why/why not?)

No; the matrix is not normal (does not commute with its transpose).

So the separate eigenspaces are not orthogonal.

(b) For the Markov matrix $A = \begin{pmatrix} .8 & .1 \\ .2 & .9 \end{pmatrix}$, I GIVE you that the eigenvalues are 1 and .7.

Give a formula for the nth power A^n.

Compute eigenvectors for 1, say $(1, 2)^T$; and for .7, say $(-1, 1)^T$.

So for $X = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ *we have* $X^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ *and $X^{-1}AX = D$ where* $D = \begin{pmatrix} 1 & 0 \\ 0 & .7 \end{pmatrix}$.

So $A = XDX^{-1}$ and hence $A^n = XD^nX^{-1} = \frac{1}{3}\begin{pmatrix} 1 + 2(.7)^n & 1 - (.7)^n \\ 2 - 2(.7)^n & 2 + (.7)^n \end{pmatrix}$.