Math 310: **Hour Exam 2** (Solutions)

Prof. S. Smith: Fri 3 Nov 1995

You must **SHOW WORK** to receive credit.

Problem 1:

(a) Are the rows of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 4 \end{pmatrix}$ a spanning set for $\mathbf{R}^3$?

NO: Maybe easiest is $\det(A) = 0$. *Or check rref(A)* $\neq$ *identity* .

(b) In polynomial space, are the "vectors" $1 + x$, $1 - x$, and $2 + x^2$ linearly independent?

YES: The polynomial equation $a(1 + x) + b(1 - x) + c(2 + x^2) = 0$ *leads to* $Ax = 0$ *where* A *is* $\begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and we see* $\det(A) = -2 \neq 0$.

Problem 2:

(a) Find a basis for the nullspace of $\begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix}$.

Compute rref(A) as $\begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *; notice 2nd and 4th variables free.*

So basis $(-3, 1, 0, 0)^T$ *and* $(-1, 0, -\frac{1}{3}, 1)^T$.

(b) Using the basis $1, 1 + x, 1 + x^2$ for the space of polynomials of degree at most 2, give the coordinates of the "vector" $1 + x + x^2$.

Quickly: $(-1, 1, 1)$ *since we see* $(-1)(1) + (1)(1 + x) + (1)(1 + x^2) = 1 + x + x^2$.

Or in detail: the polynomial equation $r(1) + s(1 + x) + t(1 + x^2) = 1 + x + x^2$ *leads to the "*$Ax = b$*" problem* $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$ *which has unique solution* $(-1, 1, 1)^T$.

Problem 3: Find the matrix, in the standard bases, for the linear transformation

$$L\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 2x - y - z \\ x - 2y + z \\ -x + 3y + 2z \end{pmatrix}.$$

Apply L *to standard basis, put into columns to get* $\begin{pmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ -1 & 3 & 2 \end{pmatrix}$.

Problem 4:

(a) Find an orthonormal basis for the column space of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$.

Let's use Gram-Schmidt with normalization afterwards. *Call initial vectors* x_1,x_2.
Form $y_2 = x_2 - \frac{x_2 \cdot x_1}{x_1 \cdot x_1} x_1 = (1,1,1) - \frac{5}{9}(2,1,2) = \frac{1}{9}(-1,4,-1)$.
Now normalize to get $u_1 = \frac{1}{3}(2,1,2)$ *and* $u_2 = \frac{1}{\sqrt{18}}(-1,4,-1)$.

(b) Now use part (a) to give the QR factorization of A.
(That is, $A = QR$ where Q has orthonormal columns and R is upper triangular).

We find $r_{11} = \| x_1 \| = 3$, $r_{12} = u_1^T x_2 = \frac{5}{3}$, $r_{22} = \| y_2 \| = \sqrt{\frac{18}{81}} = \frac{\sqrt{2}}{3}$.

So $Q = \begin{pmatrix} \frac{2}{3} & -\frac{1}{\sqrt{18}} \\ \frac{1}{3} & \frac{4}{\sqrt{18}} \\ \frac{2}{3} & -\frac{1}{\sqrt{18}} \end{pmatrix}$ *and* $R = \begin{pmatrix} 3 & \frac{5}{3} \\ 0 & \frac{\sqrt{2}}{3} \end{pmatrix}$.

Problem 5: Find the least-squares approximate-solution of the inconsistent system

$$\begin{pmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

What is the error with this approximation?

Multiply A^T *by the augmented matrix* $[A|b]$ *to get*

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \end{pmatrix} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -3 & 1 \\ 0 & 0 & 2 \end{array}\right) = \left(\begin{array}{cc|c} 5 & -5 & 5 \\ -5 & 10 & 0 \end{array}\right).$$

Row operation $A_2^{1\times 1}$ *leads to* $\left(\begin{array}{cc|c} 5 & -5 & 5 \\ 0 & 5 & 5 \end{array}\right)$. *This gives* $y = 1$ *and then* $x = 2$.

Multiply by A *to get approximation* $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, *so error is* $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$, *of length* 2.