Math 310: **Hour Exam 2** (Solutions)

Prof. S. Smith: Mon 1 Nov 1993

Problem 1: Let W be the subspace of $\mathbf{R}^3$ spanned by the vectors $(\ 1\ 1\ 1\ )$ and $(\ 1\ 2\ 3\ )$. Find the orthogonal space $W^\perp$, and the matrix P of projection onto $W^\perp$.

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \overset{A_2^{-1\times 1}}{\rightarrow} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ *gives special solution* $a = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$; *its span gives* $W^\perp$.

So P *is* $a\frac{1}{a^Ta}a^T = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \frac{1}{6}\ (\ 1\ \text{-}2\ 1\ )\ = \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$.

Problem 2: Solve the least-squares problem for the system $\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$.

Multiply by A^T *to get normal equations* $\left(\begin{array}{cc|c} 3 & 6 & 10 \\ 6 & 14 & 25 \end{array}\right) \overset{A_2^{-2\times 1}}{\rightarrow} \left(\begin{array}{cc|c} 3 & 6 & 10 \\ 0 & 2 & 5 \end{array}\right)$.

Back-solve to get $y = \frac{5}{2}$ *and then* $x = -\frac{5}{3}$; *that is,* $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -10 \\ 15 \end{pmatrix}$.

Multiply by A *to get* $\frac{1}{6}\begin{pmatrix} 5 \\ 20 \\ 35 \end{pmatrix}$ *as approximation to original* $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$.

Problem 3: Apply the Gram-Schmidt process to the columns of $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}$.

Use the result to find the QR-factorization.

Call the columns a, b, c; *begin by setting* $A = a$.

Then $B = b - proj_A(b) = b - A\frac{1}{A^TA}(A^Tb) = b - A\cdot\frac{1}{2}\cdot 1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Now note that c *was orthogonal to* a, b; *hence to "adjusted"* A, B.
So $C = c - proj_A(c) - proj_B(c)$ *gives just* c *since projections are 0.*

Divide A, B, C *by lengths to get columns of orthogonal matrix* $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$.

Transpose columns and multiply by a, b, c *above diagonal to get* $R = \begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{3}{\sqrt{6}} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$.

Problem 4: Compute the determinant $|A|$ for $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$;
first using row operations, then by the cofactor method.

(row ops) $\xrightarrow{A_2^{-1\times 1}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{A_3^{-1\times 2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$ *so* $|A| = 1 \cdot 1 \cdot 2 = 2.$

(cofactors, top row) $|A| = 1(1\cdot 1 - 1 \cdot 0) - 0 + 1(1 \cdot 1 - 0 \cdot 1) = 1 + 1 = 2.$

Problem 5: Use Cramer's rule to solve $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$|A| = 2 \cdot 4 - 1 \cdot 3 = 5.$ *So* $x = \frac{1}{5} \begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} = 2$ *And* $y = \frac{1}{5} \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -1$.

Problem M: (Makeup from Exam 1)

Show that the set W of symmetric 2x2 matrices (those of form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$)
is a subspace of the space of all 2x2 matrices.

Note that "symmetric" requires the same value ("b") in the two off-diagonal positions.

Add two such matrices: $\begin{pmatrix} a & b \\ b & c \end{pmatrix} + \begin{pmatrix} d & e \\ e & f \end{pmatrix} = \begin{pmatrix} a+d & b+e \\ b+e & c+f \end{pmatrix}$.

Same value $b + e$ *off diagonal, so the sum is also symmetric.*

Multiply such a matrix by a scalar r*:* $r \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} ra & rb \\ rb & rc \end{pmatrix}$.

Same value rb *off diagonal, so the product is also symmetric.*

So W *is closed under both operations, hence forms a subspace.*