Math 310: **Final Exam** (Solutions)

Prof. S. Smith: Wed 8 Dec 1993

Problem 1:

(a) Find the eigenvalues of $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(b) Is A diagonalizable? (Give reason, not necessary to *do* diagonalization).

(c) Is A positive definite? Give *two* methods of deciding.

(a) We see $\det(A - xI) = (3-x)[(1-x)^2 - 2\cdot 2] = (3-x)(x^2 - 2x - 3) = (3-x)(x-3)(x+1)$.
So eigenvalues $-1, 3, 3$.

(b) Yes. Get linearly independent eigenvectors $(1,1,0)$ *and* $(0,0,1)$ *for 3.*
So with eigenvector $(1,-1,0)$ *for* -1, *have basis of eigenvectors.*

(c) No. From above, not eigenvalues are positive.
Alternatively, the three "upper left" determinants are $1, -3, -9$, *not all positive.*

Problem 2: Let a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by

$$T(v_1, v_2, v_3) = (3v_1 + 2v_2 + v_3, 2v_1 + v_2, v_2).$$

Give the matrix (in the *standard* basis) for T.

Compute $T(1,0,0) = (3,2,0)$ *and* $T(0,1,0) = (2,1,1)$ *and* $T(0,0,1) = (1,0,0)$.

Use as columns to get $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Problem 3: Let M be the Markov matrix $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

(a) Find the "steady state" eigenvector for M. (Components of vector should add to 1).

(b) Diagonalize M and use this to get an expression for the power M^n.

(a) For eigenvalue 1, $M - 1I = \frac{1}{3}\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$,
so eigenvector is span of $(1,1)^T$. *Hence* $(\frac{1}{2}, \frac{1}{2})$ *is steady-state vector.*

(b) Compute $\det(M - xI) = (x - \frac{1}{3})^2 - \frac{2}{3}^2 = x^2 - \frac{2}{3}x - \frac{1}{3} = (x-1)(x+\frac{1}{3})$.

So other eigenvalue is $-\frac{1}{3}$; *from* $M + \frac{1}{3}I = \frac{1}{3}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ *get eigenvector* $(1,-1)^T$.

Using $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, *get* $S^{-1}MS = \Lambda$ *so* $M = S\Lambda S^{-1}$ *hence* $M^n = S(\Lambda^n)S^{-1}$.

Thus $M^n = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 \\ 0 & (-\frac{1}{3})^n \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (-\frac{1}{3})^n \\ 1 & -(-\frac{1}{3})^n \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + (-\frac{1}{3})^n & 1 - (-\frac{1}{3})^n \\ 1 - (-\frac{1}{3})^n & 1 + (-\frac{1}{3})^n \end{pmatrix}$.

Problem 4: Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. I *give* you that the eigenvalues are $3, 0, 0$.

(a) Find eigenvectors for these eigenvalues.

(b) Note A is *symmetric*. So find *orthogonal* S with $S^{-1}AS$ diagonal.

(Remember this means the columns of S must be *orthonormal*).

(c) Give the matrix P for projection on your 3-eigenvector.

(a) $A - 3I$ row-reduces to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, *so a 3-eigenvector is* $(1,1,1)^T$.

$A - 0I = A$ *row-reduces to* $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

so independent 0-eigenvectors are $(1,-1,0)^T$ *and* $(1,0,-1)^T$.

(b) We need to apply Gram-Schmidt to each eigenspace.

For 3, just divide by length to get $\frac{1}{\sqrt{3}}(1,1,1)^T$.

For 0, first is $(1,-1,0)^T$ *and second is (column form of)*

$$(1,0,-1) - (1,-1,0)\frac{1}{(1,-1,0)(1,-1,0)^T}(1,-1,0)(1,0,-1)^T$$

$$= (1,0,-1) - \tfrac{1}{2}(1,-1,0) = \tfrac{1}{2}(1,1,-2).$$

Divide by lengths to get $\frac{1}{\sqrt{2}}(1,-1,0)^T$ *and* $\frac{1}{\sqrt{6}}(1,1,-2)^T$.

So use $S = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}$.

(c) So $P = (1,1,1)\frac{1}{(1,1,1)(1,1,1)^T}(1,1,1)^T = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.