Michael Aschbacher's work and the Classification of Finite Simple Groups

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Schock Prize Symposium
Stockholm, 3 November 2011

Overview of the talk:

0: The "Dichotomy" Theorem—determining the Odd and Even Cases in the CFSG.
1: The Odd Case: component type; handled via "standard form".
2: The Even Case: characteristic 2 type; handled via a TRIchotomy of subcases.
∞: Further remarks on Aschbacher's many fundamental ideas and results.

slides at: `www.math.uic.edu/~smiths/talk.pdf`

Preface: an EXPOSITION of the CFSG

Gorenstein began an outline-project,
–with a "preliminaries" volume [Gor82]:

(subtitle: An introduction to their classification)
–and an Odd Case overview in [Gor83]:

(vol 1: Groups of noncharacteristic 2-type)
But the Even Case CFSG proof was incomplete...
...until the quasithin subcase was classified in [AS04].

And now the Even Case of Gorenstein's outline project has also been completed, with [ALSS11]:

("vol 2": Groups of characteristic 2 type)
by Aschbacher, Lyons, Smith, and Solomon.

(The outline in this talk largely follows [ALSS11].)

§0: Dichotomy–into Odd and Even Cases

§§0.1: How to START toward CFSG?

A famous result distinguishes odd vs even PRIMES:

Theorem(Feit-Thompson 1963):
A group of odd order is solvable.

So our simple group G must have *even* order...
...hence contains some t of order 2 (involution).
Now t commutes with its centralizer $C_G(t)$...
...so the latter is not simple—hence $C_G(t) < G$.
If G is a minimal counterexample to the CFSG,
then simple sections of smaller $C_G(t)$ are KNOWN.

Brauer proposed such an approach to the CFSG:
–Consider each H with a central involution;
–Determine simple G having t with $C_G(t) \cong H$.
In that direction: Brauer-Fowler (1955) showed
that each H can lead to only finitely many G.

What STRUCTURES in $C_G(t)$ might we exploit?
Let's "cheat" by looking ahead to the ANSWER:
CFSG Theorem: A simple group is one of:
–An alternating group A_n ($n \geq 5$);
–A Lie-type group (matrix groups over finite $\mathbb{F}_q$);
–one of 26 "sporadic" groups.

§§0.2: Odd/even $C_G(t)$ in matrix groups

Example: $G := GL_n(q)$—it's "nearly" simple...

odd q

$$t: \left(\begin{array}{ccc|ccc} -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \\ \hline & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{array}\right) \qquad C_G(t): \left(\begin{array}{c|c} GL_k & 0 \\ \hline 0 & GL_{n-k} \end{array}\right)$$

component: quasisimple subnormal subgroup;

component TYPE: For some t and $C_t := C_G(t)$, there is a component in $\overline{C}_t := C_t/O_{2'}(C_t)$.

even q

$$t: \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ 1 & & & & 1 \end{pmatrix} \qquad C_t: \left(\begin{array}{c|ccc|c} * & & 0 & & 0 \\ \hline * & & & & \\ * & & * & & 0 \\ * & & & & \\ \hline * & * & * & * & * \end{array}\right)$$

Here GL_{n-2} is NOT a component (not normal);

...indeed ONLY $O_2(C_t)$ contains normal subgroups.

characteristic 2 TYPE: Each t has $F^*(C_t) = O_2(C_t)$.

§§0.3: PROVING Odd/Even Dichotomy

Early results handled "small" cases—$m_2(G) \leq 2$.

Dichotomy Theorem: If $m_2(G) \geq 3$, then
either G has component type,
or G has characteristic 2 type.

Proof: (Sketch!) Assume component type fails.
Then each t has $F^*(\overline{C}_t) = O_2(\overline{C}_t)$.
NEED this for C_t—so suffices to get $O_{2'}(C_t) = 1$.

Method: Study the function $\theta : t \mapsto O_{2'}(C_t)$;
and show $\theta \equiv 1$ (only TRIVIAL images).

Balance leading to signalizer functor:

The absence of components gives "balance" :

$$\theta(t) \cap C_u \leq \theta(u) \cap C_t, \text{ for } t \in C_u.$$

It follows that θ, on larger 2-groups, is a functOR:
i.e., is natural with respect to $\leq$ and conjugacy.

Completeness and connected components:

As $m_2(G) \geq 2$, quote Signalizer Functor Theorem:
—θ must be "complete", for each A of rank ≥ 3;
—includes $\theta(B) = \theta(A)$ for $B \leq A$ of rank ≥ 2.
It follows θ is constant on components of graph Γ:
with vertices like A, edges by intersecting in a B.

The DISconnected case:

Here, we quote the Strongly Embedded Theorem:
G is of Lie type and rank 1, in characteristic 2.
Such G DO have characteristic 2 type (and $\theta \equiv 1$).

The connected case:

Here, $\theta(A)$ is constant on the conjugacy class of A;
so that G normalizes $\theta(A)$.
By simplicity, $\theta(A) = 1$; hence $\theta(t) = 1$ for $t \in A$.
That is, $O_{2'}(C_t) = 1$, for any involution t.
Recall this sufficed here to give $F^*(C_t) = O_2(C_t)$.
Thus connecteness also gives characteristic 2 type,
completing our proof of the Dichotomy Theorem.

The notions of balance and signalizer functors,
with completeness and (dis-)connectedness
leading to dichotomies (indeed TRIchotomies),
were extended in many ways, throughout the CFSG.

§1: The Odd Case—component type, handled via standard form

We outline via a 6-step STRATEGY:

First, we recall some preliminaries to Dichotomy...

STEP 1: Small cases–$m_2(G) \leq 2$.

(Burnside) Brauer-Suzuki Gorenstein-Walter Alperin-Brauer-Gorenstein Lyons...

STEP 2: Disconnected ("uniqueness")

Bender-Suzuki 1971 covered "strongly embedded".

(Digression: VARIATIONS on disconnectedness:

—Gorenstein-Harada: sectional 2-rank ≥ 4.

—Aschbacher: "Proper 2-generated core")

We saw Dichotomy; now ASSUME the Odd Case:

STEP 3: We have a component $\overline{L}$ in $\overline{C}_t$.

Here G should be Lie type/odd + alternating;

...so we EXPECT components L of those types.

Aschbacher idea: An L suitably MAXIMAL in C_t should have "standard form" restrictions on C_t.

First, need to "pull back" from $\overline{C}_t$ to C_t:

STEP 4: Get a component L in C_t.

Aschbacher originally ASSUMED Thompson's
B-Conjecture—was PROVED, so B**-Theorem**:
If $O_{2'}(G) = 1$, preimage$(\overline{L})$ is a component L.

...obtained via **Unbalanced Group Theorem**:
If $O_{2'}(C_t) \neq 1$, then G is (Lie(odd), A_n, ...).

Proof? used "unbalancing" possibilities $\overline{L}$;
with a notion of standardness for 2-components.
—(Solomon) $\overline{L} \cong 2A_n (n \geq 8)$: get few G.
—(Aschbacher) **Classical Involution Thm**:
$\overline{L} \cong SL_2(q)$ or $2A_7$: G is Lie(odd) (or M_{11}).
Thompson reduced to $\overline{L} \cong L_2(q)$, A_n, (+few).
—(Solomon) $\overline{L} \cong A_n (n \geq 8)$: G is A_{n+4}.
—(Harris, Walter, ...) $\overline{L} \cong$ Lie(odd) (incl $L_2(q)$):
then G is Lie(odd), or A_n, or (few).
—other $\overline{L}$: Foote Harris-Solomon Fritz Griess-Solomon Aschbacher-Seitz Cheng

STEP 5: Get an L in "standard form".

Using the B-Theorem, we get the FINAL form of Aschbacher's **Standard Component Theorem**:

For G of component type with $m_2(G) \geq 3$, (either G has classical involution, so known, or) some C_t has an L which is STANDARD in G.

That is, L does not commute with its conjugates; and $C_G(L)$ is TIGHTLY embedded.

STEP 6: For each L, find possible G.

...expect G is USUALLY of "same type" as L.

MANY authors did "standard form problems".

So, just a few remarks here:

—L Lie(odd) or A_n? done as $\overline{L}$ in Step 4.

—Aschbacher-Seitz did case $m_2(C_G(L)) \geq 2$.

—Seitz reduced to L of small Lie rank.

—Many L done by Finkelstein, Solomon, Seitz

§2: The Even Case—characteristic-2 type, handled in a Trichotomy subdivision

Now, assume the REMAINING case from Dichotomy;
...namely G is of characteristic 2 type.
(That is, $F^*(N) = O_2(N)$ for each 2-local N.)
GOAL: Mimic standard- L Strategy—"for odd p";
i.e. replace 2: now $C_G(u)$ for u of order p, etc.

We can START much as in the Odd Case:

STEP 1: Small case–$e(G) \leq 2$ (quasithin)

"Size" is measured by Thompson's early notion:
$e(G) = \max m_p(N)$ over odd p, 2-locals N.
Aschbacher 1978 handled $e(G) = 1$ ("thin").
Mason did NOT publish $e(G) \leq 2$ ("quasithin").

Finally, Aschbacher-Smith [AS04] classified:
quasithin simple G of "even characteristic"
i.e. $F^*(N) = O_2(N)$ just for $N \geq$ 2-Sylow of G.

Notoriously lengthy; so here only a few remarks:
—a STRUCTURE THEORY; "$\mathcal{C}$-components" L.
"Thompson strategy" for 2-locals M and H:
—list of possible maximal L (and $V \leq O_2(L)$).
—list of H with $O_2(H) \neq 1$, min'l $H \not\leq M$.
(Courage!) For each pair, find $< M, H >= G$.

So now, we may assume "generic" case $e(G) \geq 3$.
We get a VARIATION from the earlier Strategy:

STEP 2: p-Uniqueness—postponed!

In contrast to Strongly (2-)Embedded,
NO treatment of strongly p-embedded is known,
except as a consequence of the full CFSG.
So that situation is known only inductively;
...and then "p-uniqueness" is treated at the END.

Modified STRATEGY: via a trichotomy.

The Dichotomy argument (for 2) also applies to p;
with "disconnected", we get a (weak) TRIchotomy.
Can even get **(Strong) Trichotomy Theorem:**
($e(G) \geq 4$, Gorenstein-Lyons; $= 3$, Aschbacher)
One of the following holds for G:
(1) p-component type (leading to standard TYPE);
(2) p-disconnected (leading to p-Uniqueness Case);
(3) characteristic p type (leading to $GF(2)$ type).

(Proof? generalize Dichotomy arguments;
...DO Steps 3–5 of old-Strategy, in getting (1).)

It remained to treat the Trichotomy cases (1)–(3).

Case (3)—$GF(2)$ type: ALREADY done!

Not "essential": Klinger-Mason 1975 showed that char-$\{2, p\}$-type does NOT arise when $e(G) \geq 3$.

But for Gorenstein-Lyons, it was CONVENIENT to use "similar" arguments to lead to sporadic G. (Leaving standard type to lead to Lie(even) G.)

Namely "intermediate" Klinger-Mason arguments lead to G having "$GF(2)$" type: some involution t has $O_2(C_t)$ of symplectic type.

Then they QUOTE the $GF(2)$ type classification; already done (independently of CFSG) by 1978.
—work of Aschbacher, Timmesfeld, Smith ...
—GENERAL answer: mainly, Lie type over $\mathbb{F}_2$;
HERE: get all sporadic G-L conclusions.
—(There is an extension, classifying $GF(2^n)$ type.)

Case (1): Handling Standard Type

(This corresponds to Step 6 in old-Strategy)

Standard TYPE: more specific than standard form.

...for example, "neighboring" components (intersecting in a sub-component, standard in each).

The ANSWER has includes all G which are Lie(even).

In contrast to many standard form papers in §1, standard type done in ONE: by Gilman-Griess.

(Some $e(G) = 3$ cases by Finkelstein-Frohardt.)

Recognition of Lie type G—via Steinberg relations, and Curtis-Tits (+ Phan) presentations.

Case (2): Handling the Uniqueness Case

(This is "postponed Step 2" of old-Strategy)

Aschbacher showed that NO group G arises here; so this is the final contradiction in the CFSG.

(Roughly, a strongly p-embedded subgroup should be a p-local—and NOT a 2-local.)

(With G-L:) pre-uniqueness $\Rightarrow$ Uniqueness Case.

Thompson strategy: maxl 2-local M; $H \not\leq M$.

—weak closure: get $H = H_1 H_2$, for $H_i \leq N_G(A_i)$;

—uniqueness thms: $\exists!$ maxl 2-local over $N_M(A_i)$.

That local is M: $\geq N_G(A_i) \geq H_i$; $\rightarrow\leftarrow$ $H \not\leq M$.

§∞: Some further remarks on Aschbacher's ideas and results

Here at least is a (partial) LIST of contributions:

(General Techniques:)

Characterizations ("recognition") of various groups
Odd transpositions; groups of GF(2) type ...
Proper 2-Generated Core Theorem
Failure of factorization theory
Pushing up; blocks and the $C(G,T)$-Theorem
Tightly embedded subgroups
Involutions in Lie type in char 2 (with Seitz)
Weak closure
Uniqueness theorems

(Odd Case:)

Classical Involution Theorem
Standard Form Theorem
Reductions/solutions for standard form problems

(Even Case:)

Quasithin classification (with Smith)
$e(G) = 3$ Classification (... via Trichotomy)
Uniqueness Case (+ preuniqueness, with G-L)

(Outside the CFSG:)

Geometric structures for simple groups
Maximal subgroups (with Scott)
Uniqueness of sporadic groups (with Segev)
On Quillen's conjecture for $\mathcal{S}_p(G)$ (with Smith)
Fusion systems and simple groups

...and did I mention his influential BOOKS ?

Epilogue: Later approaches to the CFSG

Gorenstein-Lyons-Solomon project ("revisionism")
—"second effort": more methodical...

Meierfrankenfeld-Stellmacher approach
—organized around characteristic p type ...

(early) Aschbacher: via simple fusion systems?

References

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