Research Statement: Jeff Sommars

Computational Tropical Geometry and Applications

My research interests lie in computational tropical geometry, specifically in developing algorithms and software packages to perform tropical computations. *Tropical geometry* is a combinatorial shadow of algebraic geometry; in it, problems in algebraic geometry are transformed into polyhedral problems. As a graduate student, the computations I was most interested in were motivated by connections between tropical geometry and numerical algebraic geometry. For my postdoctoral research, I propose a series of projects from computational tropical geometry: two that continue my current research program and two that extend a new tropical application.

Background

Tropical Geometry

Let $F(\boldsymbol{x})$ be a polynomial system in $\mathbb{C}[x_1, x_2, \ldots, x_n]$ with m polynomials, and let I be its associated ideal. The *Newton polytope* $\mathrm{NP}(f)$ of a polynomial $f \in F$ is the convex hull of the exponent vectors appearing with nonzero coefficient in f. Each face of $\mathrm{NP}(f)$ has a normal cone. The set of all normal cones constitutes the *normal fan* of $\mathrm{NP}(f)$, which is a polyhedral fan in $\mathbb{R}^n$. A *tropical hypersurface* $\mathrm{T}(f)$ is the set of codimension one cones in the normal fan of $\mathrm{NP}(f)$, or equivalently it is the set

$$\mathcal{T}(f) = \{w \in \mathbb{R}^n : \mathrm{in}_w(f) \text{ is not a monomial}\}. \tag{1}$$

A *tropical prevariety* of a polynomial system is defined to be $\bigcap_{f \in F} \mathcal{T}(f)$. The tropical prevariety is a combinatorial object, depending only on the Newton polytopes of the polynomials in the system, whereas the cancellation properties of the coefficients are captured by the *tropical variety* of I, $\mathcal{T}(I)$. A tropical variety is the intersection of all possible tropical hypersurfaces $\mathcal{T}(f)$ for each $f \in I$. In [6], it is proven that every tropical variety is also a tropical prevariety, that is, there is some finite set of tropical hypersurfaces from polynomials in I that can be intersected to retrieve $\mathcal{T}(I)$.

Tropical Auction Theory

During the 2007 financial crisis, the Bank of England needed to auction goods quickly to provide liquidity to banks. However, the speed of financial markets necessitated that these auctions happen instantaneously, instead of in a multi-round auction. In response to this need, Klemperer designed the *product-mix auction* [19]. The Bank of England used it regularly in 2007-2008 and they have used it again in the aftermath of Brexit [20].

In economic terms, a product-mix auction is a single round, sealed bid auction to sell differentiated goods. This means that all goods are priced and sold in a single round without any bidder knowing any other bidder's bid. To achieve this result, bidders submit tropical hypersurfaces that approximate their demand functions. To compute the equilibrium price vector of the auction, the auctioneer performs a series of tropical computations, including computing tropical prevarieties and mixed volumes([11], [26]). This was the first connection between tropical geometry and economics, but new ones continute to develop (for example, [8] and [18]).

Past Accomplishments

As part of my Ph.D. research, I developed new algorithms for computing tropical prevarieties ([24] and [16]). I implemented the most efficient of these [16] in the software package *DynamicPrevariety* (available at [23]).

This software was the first to compute the tropical prevariety of the cyclic-16 roots problem, a specific instance of the cyclic-n roots problem:

$$\begin{cases} x_0 + x_1 + \cdots + x_{n-1} = 0 \\ i = 2, 3, \ldots, n-1 : \displaystyle\sum_{j=0}^{n-1} \prod_{k=j}^{j+i-1} x_{k \bmod n} = 0 \\ x_0 x_1 x_2 \cdots x_{n-1} - 1 = 0. \end{cases} \tag{2}$$

Cyclic-16 is a challenging benchmark problem used throughout polynomial system solving, for example, in [13] and [21]. In a different computation, Yue Ren and I used my software to disprove the conjecture of Q. Ren, Shaw, and Sturmfels in [22]. The main step in disproving this conjecture was computing a tropical prevariety of 270 polynomials that Gfan could not compute. *DynamicPrevariety* finished the computation in under thirty seconds on an ordinary laptop.

I have also collaborated on developing a new numerical framework for solving polynomial systems [10]. This approach uses homotopy continuation and the monodromy action to find the solution set of a generic system in a family of polynomial systems. Our implementation of this algorithm [9] is competitive with the existing state-of-the-art methods implemented in other software packages. In certain cases, the method performs far better than other algorithms, for example [7]. We are in the process of developing a parallel implementation of this algorithm; preliminary results show that we achieve near linear speedups [4].

Proposed Research

I propose two general areas of research in computational tropical geometry:

- explore the relationship between tropical prevarieties and tropical varieties, and
- expand the usefulness of the product-mix auction.

Tropical (Pre)varieties

These projects aim to develop a deeper understanding of the relationship between tropical prevarieties and tropical varieties. Through performing these projects, I hope to gain new insight into how these tropical constructs can be used to inform numerical algebraic geometry. For the following two projects, let $\boldsymbol{x} = x_1, x_2, \ldots, x_n$, let $f(\boldsymbol{x})$ be a polynomial system with a non-trivial tropical prevariety of dimension d, and let I be the ideal generated by the polynomials in $f(\boldsymbol{x})$.

Project 1 *Determine for each $i \in \{0, 1, \ldots, d\}$ if it is possible to choose non-zero coefficients for $f(\boldsymbol{x})$ such that dim($\mathcal{T}(I)$) = i. If so, find a point in the parameter space such that dim($\mathcal{T}(I)$) = i.*

To develop a strategy for an algorithmic solution, I will experiment with the cyclic n-roots polynomial system (2). In [2], Backelin showed that if n has a divisor that is a square, i.e. if d^2 divides n for $d \geq 2$, then there are infinitely many cyclic n-roots, including a $d-1$ dimensional component. The conjecture of Björck and Saffari [3], [12, Conjecture 1.1] is that if n is not divisible by a square, then the set of cyclic n-roots is finite.

I will begin with the cyclic 8-roots problem, as it is sufficiently large to have an interesting tropical prevariety, but not too large for Gröbner bases to be computationally infeasible. The cyclic 8-roots problem has a tropical prevariety with 48 three dimensional maximal cones, 16 two dimensional maximal cones and 32 one dimensional maximal cones. From Backelin's lemma, if all coefficients are 1, then dim $\mathcal{T}(I) = 1$, as also demonstrated in [1]. Furthermore, if the coefficients are generic, dim $\mathcal{T}(I) = 0$ as cyclic 8 is a square system. However, it is not clear how to pick coefficients such that the tropical variety is either two or three dimensional, or even if it is possible to pick the coefficients in such a way. I believe that by exploring the parameter space while computing tropical varieties, it will be possible to find some regions where dim $\mathcal{T}(I) > 1$. Through this experiment, I will develop ideas for a general algorithm that will apply to any polynomial system.

Project 2 *Develop a probabilistic method for computing* $\mathcal{T}(I)$ *from* $\bigcap_{f \in F} \mathcal{T}(f)$.

Computing a tropical variety is possible through the use of software packages like Gfan [17], but to compute one symbolically requires computing many Gröbner bases. An algorithm has been made to compute them numerically [14], but so far, there is no software that can consistently perform the computations.

It seemed possible that $\mathcal{T}(I)$ was equal to the tropical prevariety with some number of cones removed from it. However, this was demonstrated not to be true in [5]: there can be cones of $\mathcal{T}(I)$ strictly contained within cones of the tropical prevariety (see Figure 1 for an example). If this had been true, it would have sufficed to test if interior points in maximal cones of the tropical prevariety were in the tropical variety.

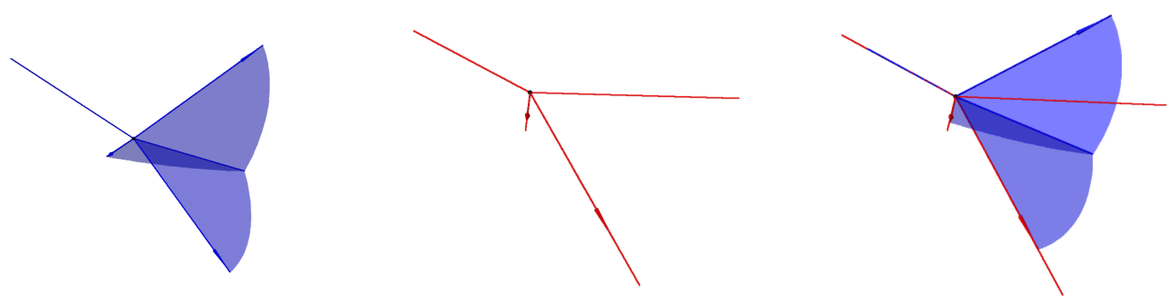

Figure 1: Let $I = \{x_1^2 - x_1 + x_2 + x_3, x_2^2 + x_1 + x_2 + x_3\}$. From left to right: the tropical prevariety of I, the tropical variety of I, and both. There is a one dimensional maximal cone of the tropical variety hidden inside a maximal cone of the tropical prevariety.

To deal with these computational challenges, I will develop a probabilistic algorithm to determine if individual cones of the tropical prevariety are in the tropical variety. A naive attempt would be to randomly generate additional elements in I, compute the tropical prevariety of this larger set of polynomials, and then assume that this is equal to the tropical variety. The flaws with this approach are clear: there is no correct way to choose elements in the ideal, and it cannot be known when sufficiently many elements in the ideal have been chosen. This project aims to find a better way than this to probabilistically compute a tropical variety, perhaps through considering random projections as in [15].

Tropical Auction Theory

Project 3 *Modify the product-mix auction so that it can be applied in an online setting.*

For an online product-mix auction to be feasible, it is necessary that the auction integrate the results of previous auctions into the current auction. If previous auctions are ignored, bidders would be able to artificially move the prices of goods between auctions, leading to arbitrage opportunites in a secondary market. The goal of an online product-mix auction is to find a price vector such that the prices paid are fair to current bidders as well as previous bidders. Because of this, the result of the online product-mix auction may no longer be a competitive equilibrium, but the output will be a set of prices that is equitable to both current and past participants. I now describe two candidate ways of doing this task.

Option one is to expand the current product-mix auction so that it takes in weighted tropical hypersurfaces as inputs. The auction will use the tropical hypersurfaces of the previous n auctions, in addition to the current auction. Weights are assigned to hypersurfaces in accordance with their recency, giving lower weights to older tropical hypersurfaces. The algorithm follows the same general steps as the original product-mix auction, but every step requires minor modifications and is more computationally intensive.

Option two begins by performing an original product-mix auction with only the tropical hypersurfaces associated to the current auction. This determines a price vector, which I call a *computed price vector* p_c, that ignores earlier auctions. After finding p_c, it is necessary to find the *actual price vector* p_a that will be paid by current auction participants; it will be computed using p_c and the p_a of previous auctions. For $i > 0$, let p_{a_i} be the actual price vector i auctions in the past. Let $\lambda_i \in \mathbb{R} \cap (0, 1)$; if $i < j$, let $\lambda_i > \lambda_j$. The current actual price vector will be $p_a = \lambda_0 p_c + \lambda_1 p_{a_1} + \lambda_2 p_{a_2} + \ldots + \lambda_n p_{a_n}$, such that $\sum_{i=1}^{n} \lambda_i = 1$. In other words, the actual price vector should be in the convex hull of the current computed price vector and the past actual price vectors; the task is to find an optimal convex combination.

There are a variety of differences between the two options. The first option will be computationally expensive, possibly requiring exponentially more tropical prevarieties and mixed volume computations than the original product-mix auction; this could make it impractical in practice. On the contrary, option two could be almost as fast to perform as the original product-mix auction. However, the more significant difference between the two options is that the first requires the scaling factors of the earlier auctions to be input by the auctioneer, while the latter allows the λ_i to float.

Project 4 *Develop a standalone software package that implements the product-mix auction.*

The final portion of this project will be developing a software package that implements the original product-mix auction, as well as the online product-mix auction. When designing this software, I will use best practices for software development. These include object-oriented design, modular design, and the use of source control. I will keep all of the development free and open source, under GPL, and I will host it on a public git repository on my GitHub page.

Computing tropical prevarieties is a crucial part of the product-mix auction, and I expect this to be true for the online product-mix auction as well. Because of this, I plan on integrating portions of my existing code base into the software package I develop. Currently, my software is optimized for computing tropical prevarieties of n tropical hypersurfaces, where n is large. For the standard product-mix auction, $n = 2$. I envision another algorithm, not yet integrated into my software, that will perform excellently in the $n = 2$ case, similar to the algorithm in [24]. However, for the online product-mix auction, I believe that both algorithms will be necessary.

Broader Impacts

These proposed projects have the potential to have real world impacts in diverse areas. The first two projects work towards developing a deeper understanding of tropical varieties. This is a component of a larger research goal: computing parametrizations of positive dimensional components of polynomial systems. It is generally understood that this can be done through computing a tropical variety and then computing multi-dimensional Puiseux series; however, there are many reasons why this is difficult in practice. The first challenge is that computing a tropical variety is prohibitively slow, but perhaps if we gain a better understanding of the relationship between tropical prevarieties and tropical varieties, we will discover practical ways to circumvent this difficulty. Parametrizing solution sets of polynomial systems may make algebraic geometry more accessible to new groups of researchers, those who are more comfortable with series expansions than with algebraic varieties.

The online product-mix auction will allow goods to be dynamically priced in online auctions that used to have manual pricing. One example of this can be seen in marketplace lending, a new method for connecting individuals in need of capital with people looking to invest in debt. Traditionally, the originators of marketplace notes have priced the loans, but this can result in borrowers paying more interest than the market requires or in loans going unfunded. Auctioning marketplace notes in an online product-mix auction would eliminate this deadweight loss. Currently, more than twenty billion dollars a year of consumer debt are funded through marketplace lending in the United States alone [25], so it would be significant to improve the efficiency of this market.

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