ON CRDAHA AND FINITE GENERAL LINEAR AND UNITARY GROUPS

BHAMA SRINIVASAN

Abstract. We show a connection between Lusztig induction operators in finite general linear and unitary groups and parabolic induction in cyclotomic rational double affine Hecke algebras. Two applications are given: an explanation of a bijection result of Broué, Malle and Michel, and some results on modular decomposition numbers of finite general groups.

2010 *AMS Subject Classification:* 20C33

Dedicated to the memory of Robert Steinberg

1. Introduction

Let Γ_n be the complex reflection group $G(e,1,n)$, the wreath product of S_n and $\mathbb{Z}/e\mathbb{Z}$, where $e > 1$ is fixed for all n. Let $H(\Gamma_n)$ be the cyclotomic rational double affine Hecke algebra, or CRDAHA, associated with the complex reflection group Γ_n. The representation theory of the algebras $H(\Gamma_n)$ is related to the representation theory of the groups Γ_n, and thus to the modular representation theory of finite general linear groups $GL(n,q)$ and unitary groups $U(n,q)$. In this paper we study this connection in the context of a recent paper of Shan and Vasserot [19]. In particular we show a connection between Lusztig induction operators in general linear and unitary groups and certain operators in a Heisenberg algebra acting on a Fock space. We give two applications of this result, where ℓ is a prime not dividing q and e is the order of q mod ℓ. The first is a connection via Fock space between an induction functor in CRDAHA described in [19] and Lusztig induction, which gives an explanation for a bijection given by Broué, Malle and Michel [1] and Enguehard [6] between characters in an ℓ-block of a finite general linear, unitary or classical group and characters of a corresponding complex reflection group. The second is an application to the ℓ-modular theory of $GL(n,q)$, describing some Brauer characters by Lusztig induction, for large ℓ.

The paper is organized as follows. In Section 3 we state the results on CRDAHA from [19] that we need. We introduce the category $\mathcal{O}(\Gamma) = \oplus_{n\geq 0}\mathcal{O}(\Gamma_n)$ where $\mathcal{O}(\Gamma_n)$ is the category $\mathcal{O}$ of $H(\Gamma_n)$.

In Section 4 we describe the ℓ-block theory of $GL(n,q)$ and $U(n,q)$. The unipotent characters in a unipotent block are precisely the constituents of a Lusztig induced character from an e-split Levi subgroup. Complex reflection groups arise when considering the defect groups of the blocks.

In Section 5 we introduce the Fock space and the Heisenberg algebra, and describe the connection between parabolic induction in CRDAHA and a Heisenberg algebra action on a Fock space given in [19]. We have a Fock space $\mathcal{F}^{(s)}_{m,\ell}$ where $m,\ell > 1$ are positive integers and (s) is an ℓ-tuple of integers. In [19] a functor a^*_μ, where μ is a partition, is introduced on the Grothendieck group $[\mathcal{O}(\Gamma)]$ and is identified with an operator S_μ of a Heisenberg algebra on the above Fock space.

The case $\ell = 1$ is considered in Section 6. We consider a Fock space with a basis indexed by unipotent representations of general linear or unitary groups. We define the action of a Heisenberg algebra on this by a Lusztig induction operator $\mathcal{L}_\mu$ and prove that it can be identified with an operator S_μ defined by Leclerc and Thibon [14]. This is one of the main results of the paper. It involves using a map introduced by Farahat [7] on the characters of symmetric groups, which appears to be not widely known.

In Sections 7 and 8 we give applications of this result, using the results of Section 5. The first application is that parabolic induction a^*_μ in CRDAHA and Lusztig induction $\mathcal{L}_\mu$ on general linear or unitary groups can be regarded as operators arising from equivalent representations of the Heisenberg algebra. This gives an explanation for an observation of Broué, Malle and Michel on a bijection between Lusztig induced characters in a block of $GL(n,q)$ and $U(n,q)$ and characters of a complex reflection group arising from the defect group of the block.

The second application deals with ℓ-decomposition numbers of the unipotent characters of $GL(n,q)$ for large ℓ. Via the q-Schur algebra we can regard these numbers as arising from the coefficients of a canonical basis $G^-(\lambda)$ of Fock space, where λ runs through all partitions, in terms of the standard basis. The $G^-(\lambda)$ then express the Brauer characters of $GL(n,q)$ in terms of unipotent characters. The $G^-(\lambda)$ are also described as S_μ, and so we finally get that if $\lambda = \mu + e\alpha$ where μ' is e-regular, the Brauer character parametrized by λ is in fact a Lusztig induced generalized character.

2. Notation

$\mathcal{P}, \mathcal{P}_n, \mathcal{P}^\ell, \mathcal{P}^\ell_n$ denote the set of all partitions, the set of all partitions of $n \geq 0$, the set of all ℓ-tuples of partitions, and the set of all ℓ-tuples of partitions of integers $n_1, n_2, \dots n_\ell$ such that $\sum n_i = n$, respectively.

If $\mathcal{C}$ is an abelian category, we write $[\mathcal{C}]$ for the complexified Grothendieck group of $\mathcal{C}$.

We write $\lambda \vdash n$ if λ is a partition of $n \geq 0$. The parts of λ are denoted by $\{\lambda_1, \lambda_2, \ldots\}$. If $\lambda = \{\lambda_i\}$, $\mu = \{\mu_i\}$ are partitions, $\lambda + \mu = \{\lambda_i + \mu_i\}$ and $e\lambda = \{e\lambda_i\}$ where e is a positive integer.

3. CRDAHA, COMPLEX REFLECTION GROUPS

References for this section are [19], [?]. We use the notation of ([19], (3.3), p.967).

Let $\Gamma_n = \mu_\ell \wr \mathcal{S}_n$, where μ_ℓ is the group of ℓ-th roots of unity in $\mathbb{C}$ and $\mathcal{S}_n$ is the symmetric group of degree n, so that Γ_n is a complex reflection group. The representation category of Γ_n is denoted by $\mathrm{Rep}(\mathbb{C}\Gamma_n)$. The irreducible modules in $\mathrm{Rep}(\mathbb{C}\Gamma_n)$ are known by a classical construction and denoted by $\bar{L}_\lambda$ where $\lambda \in \mathcal{P}^\ell_n$. Let $R(\Gamma) = \oplus_{n\geq 0}[\mathrm{Rep}(\mathbb{C}\Gamma_n)]$.

Let $\mathfrak{H}$ be the reflection representation of Γ_n and $\mathfrak{H}^*$ its dual. The cyclotomic rational double affine Hecke algebra or CRDAHA associated with Γ_n is denoted by $H(\Gamma_n)$, and is the quotient of the smash product of $\mathbb{C}\Gamma_n$ and the tensor algebra of $\mathfrak{H} \oplus \mathfrak{H}^*$ by certain relations. The definition involves certain parameters (see [19], p.967) which play a role in the results we quote from [19], although we will not state them explicitly.

The category $\mathcal{O}$ of $H(\Gamma_n)$ is denoted by $\mathcal{O}(\Gamma_n)$. This is the category of $H(\Gamma_n)$-modules whose objects are finitely generated as $\mathbb{C}[\mathfrak{H}]$-modules and are $\mathfrak{H}$-locally nilpotent. Here $\mathbb{C}[\mathfrak{H}]$ is the subalgebra of $H(\Gamma_n)$ generated by $\mathfrak{H}^*$. Then $\mathcal{O}(\Gamma_n)$ is a highest weight category (see e.g. [18]) and its standard modules are denoted by Δ_λ where $\lambda \in \mathcal{P}^\ell_n$. Let $\mathcal{O}(\Gamma) = \oplus_{n\geq 0}\mathcal{O}(\Gamma_n)$. This is one of the main objects of our study.

We then have a $\mathbb{C}$-linear isomorphism spe : $[\mathrm{Rep}(\mathbb{C}\Gamma_n)] \to [\mathcal{O}(\Gamma_n)]$ given by $[\bar{L}_\lambda] \to [\Delta_\lambda]$. We will from now on consider $[\mathcal{O}(\Gamma_n)]$ instead of $[\mathrm{Rep}(\mathbb{C}\Gamma_n)]$.

Let $r, m, n \geq 0$. For n, r we have a parabolic subgroup $\Gamma_{n,r} \cong \Gamma_n \otimes \mathcal{S}_r$ of Γ_{n+r} , and there is a canonical equivalence of categories $\mathcal{O}(\Gamma_{n,r}) = \mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathcal{S}_r)$. By the work of Bezrukavnikov and Etingof [2] there are induction and restriction functors ${}^{\mathcal{O}}\mathrm{Ind}_{n,r} : \mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathcal{S}_r) \to \mathcal{O}(\Gamma_{n+r})$ and ${}^{\mathcal{O}}\mathrm{Res}_{n,r} : \mathcal{O}(\Gamma_{n+r}) \to \mathcal{O}(\Gamma_n) \otimes \mathcal{O}(\mathcal{S}_r)$.

If $\mu \vdash r$, Shan and Vasserot ([19], 5.1) have defined functors $A_{\mu,!}, A^*_\mu, A_{\mu,*}$ on $\mathcal{D}^b(\mathcal{O}(\Gamma))$.

Here we will be concerned with A^*_μ, defined as follows.

$$\begin{aligned} A^*_\mu : \mathcal{D}^b(\mathcal{O}(\Gamma_n)) &\to \mathcal{D}^b(\mathcal{O}(\Gamma_{n+mr})), \\ M &\to^{\mathcal{O}} \mathrm{Ind}_{n,mr}(M \otimes L_{m\mu}) \end{aligned} \tag{3.1}$$

Then a^*_μ is defined as the restriction to $[\mathcal{O}(\Gamma)]$ of A^*_μ.

4. Finite general linear and unitary groups

In this section we describe a connection between the block theory of $GL(n, q)$ or $U(n, q)$, and complex reflection groups. This was first observed by Broué, Malle and Michel [1] and Enguehard [6] for arbitrary finite reductive groups.

Let $G_n = GL(n, q)$ or $U(n, q)$. The unipotent characters of G_n are indexed by partitions of n. Using the description in ([1], p.45) we denote the character corresponding to $\lambda \vdash n$ of $GL(n, q)$ or the character, up to sign, corresponding to $\lambda \vdash n$ of $U(n, q)$ as in [8] by χ_λ.

Let ℓ be a prime not dividing q and e the order of q mod ℓ. The ℓ-modular representations of G_n have been studied by various authors (see e.g. [3]) since they were introduced in [8]. The partition of the unipotent characters of G_n into ℓ-blocks is described in the following theorem from [8]. This classification depends only on e, so we can refer to an ℓ-block as an e-block, e.g. in Section 7.

Theorem 4.1. *The unipotent characters χ_λ and χ_μ of G_n are in the same e-block if and only if the partitions λ and μ of n have the same e-core.*

There are subgroups of G_n called e-split Levi subgroups ([3], p.190). In the case of $G_n = GL(n, q)$ an e-split Levi subgroup L is of the form a product of smaller general linear groups over F_{q^e} and G_k with $k \leq n$. In the case of $G_n = U(n, q)$, L is of the form a product of smaller general linear groups or of smaller unitary groups over F_{q^e} and G_k with $k \leq n$. Then a pair (L, χ_λ) is an e-cuspidal pair if L is e-split of the form a product of copies of tori, all of order $q^e - 1$ in the case of $GL(n, q)$, or all of orders $q^e - 1$, $q^{2e} - 1$ or $q^{e/2} + 1$ in the case of $U(n, q)$ and G_k, where G_k has an e-cuspidal unipotent character χ_λ ([1], p.18, p.27; [6], p.42). Here a character of L is e-cuspidal if it is not a constituent of a character obtained by Lusztig induction R_M^L from a proper e-split Levi subgroup M of L.

The unipotent blocks, i.e. blocks containing unipotent characters, are classified by e-cuspidal pairs up to G_n-conjugacy. Let B be a unipotent block corresponding to (L, χ_λ). Then if $\mu \vdash n$, $\chi_\mu \in B$ if and only if $< R_L^{G_n}(\chi_\lambda), \chi_\mu > \neq 0$. As above, $R_L^{G_n}$ is Lusztig induction.

The defect group of a unipotent block is contained in $N_{G_n}(T)$ for a maximal torus T of G_n such that $N_{G_n}(T)/T$ is isomorphic to a complex reflection group $W_{G_n}(L, \lambda) = \mathbb{Z}_e \wr S_k$ for some $k \geq 1$. Thus the irreducible characters of $W_{G_n}(L, \lambda)$ are parametrized by $\mathcal{P}_k^e$.

Let B be a unipotent block of G_n and $W_{G_n}(L, \lambda)$ as above. We then have the following theorem due to Broué, Malle and Michel ([1], 3.2) and to Enguehard ([6], Theorem B).

Theorem 4.2. *(Global to Local Bijection for G_n) Let M be an e-split Levi subgroup containing L and let $W_M(L, \lambda)$ be defined as above for M. Let μ be* *a partition, and let* I_L^M *be the isometry mapping the character of* $W_M(L,\lambda)$ *parametrized by the e-quotient of* μ *to the unipotent character* χ_μ *of* M *(up to sign) which is a constituent of* $R_M^{G_n}(\lambda)$*. Then we have* $R_M^{G_n} I_L^M = I_L^{G_n} \operatorname{Ind}_{W_M(L,\lambda)}^{W_{G_n}(L,\lambda)}$.

The theorem is proved case by case for "generic groups", and thus for finite reductive groups. We have stated it only for G_n.

We state a refined version of the theorem involving CRDAHA and prove it in Section 7.

5. Heisenberg algebra, Fock space

Throughout this section we use the notation of ([19],4.2, 4.5, 4.6).

The affine Kac-Moody algebra $\widehat{\mathfrak{sl}_\ell}$ is generated by elements $e_p, f_p, p = 0, \dots \ell-1$, satisfying Serre relations ([19], 3.4). We have $\widehat{\mathfrak{sl}_\ell} = \mathfrak{sl}_\ell \otimes \mathbb{C}[t,t^{-1}]$.

The Heisenberg algebra is the Lie algebra $\mathfrak{H}$ generated by $1, b_r, b'_r,\ r \geq 0$ with relations $[b'_r, b'_s] = [b_r, b_s] = 0$, $[b'_r, b_s] = r1\delta_{r,s}$, $r, s \geq 0$ ([19], 4.2). In $U(\mathfrak{H})$ we then have elements $b_{r_1} b_{r_2} \dots$ with $\sum_i r_i = r$. If $\lambda \in \mathcal{P}$ we then have the element $b_\lambda = b_{\lambda_1} b_{\lambda_2} \dots$, and then for any symmetric function f the element $b_f = \sum_{\lambda \in \mathcal{P}} z_\lambda^{-1} < P_\lambda, f > b_\lambda$. Here P_λ is a power sum symmetric function and $z_\lambda = \prod_i i^{m_i} m_i!$ where m_i is the number of parts of λ equal to i. The scalar product $<,>$ is the one used in symmetric functions, where the Schur functions form an orthonormal basis (see[16]).

We now define Fock spaces $\mathcal{F}_m$, $\mathcal{F}_{m,\ell}^{(d)}$ and $\mathcal{F}_{m,\ell}^{(s)}$, where $m > 1$. We first define the Fock space, $\mathcal{F}_m$ of rank m and level 1 ([19], 4.5). Choose a basis $(\epsilon_1, \dots \epsilon_m)$ of $\mathbb{C}^m$. The Fock space $\mathcal{F}_m$ is defined as the space of semi-infinite wedges of the $\mathbb{C}$-vector space $\mathbb{C}^m \otimes \mathbb{C}[t,t^{-1}]$. If $d \in \mathbb{Z}$, let $\mathcal{F}_m^{(d)}$ be the subspace of elements of the form $u_{i_1} \wedge u_{i_2} \dots$, $i_1 > i_2 \dots$, where $u_{i-jm} = \epsilon_i \otimes t^j$, $i_k = d - k + 1$ for $k \gg 0$. Then $\mathcal{F}_m = \oplus_{d \in \mathbb{Z}} {\mathcal{F}_m}^{(d)}$. If we set $|\lambda, d >= u_{i_1} \wedge u_{i_2} \dots$, $i_k = \lambda_k + d - k + 1$, the elements $|\lambda, d >$ with $\lambda \in \mathcal{P}$ form a basis of ${\mathcal{F}_m}^{(d)}$. Then $\widehat{\mathfrak{sl}_m}$ acts on ${\mathcal{F}_m}^{(d)}$. This set-up has been studied by Leclerc and Thibon ([14], [15]).

Similarly choose a basis $(\epsilon_1, \dots \epsilon_m)$ of $\mathbb{C}^m$ and a basis $({\epsilon_1}', \dots {\epsilon_\ell}')$ of $\mathbb{C}^\ell$. The Fock space $\mathcal{F}_{m,\ell}$ of rank m and level ℓ is defined as the space of semi-infinite wedges, i.e. elements of the form $u_{i_1} \wedge u_{i_1} \dots$, $i_1, i_2 \dots$, where the u_j are vectors in a $\mathbb{C}$-vector space $\mathbb{C}^m \otimes \mathbb{C}^\ell \otimes \mathbb{C}[z, z^{-1}]$ given by $u_{i+(j-1)m-km\ell} = \epsilon_i \otimes \epsilon_j^{'} \otimes z^k$, with $i = 1, 2, \dots m$, $j = 1, 2, \dots \ell$, $k \in \mathbb{Z}$. Then $\widehat{\mathfrak{sl}_\ell}$, $\widehat{\mathfrak{sl}_m}$ and $\mathfrak{H}$ act on the space ([19], 4.6).

Let $d \in \mathbb{Z}$. We have a decomposition $\mathcal{F}_{m,\ell} = \oplus_{d \in \mathbb{Z}} \mathcal{F}_{m,\ell}^{(d)}$ defined using semi-infinite wedges, as in the case of $\mathcal{F}_m$. Then $\mathcal{F}_{m,\ell}^{(d)}$ can be identified with the space $\Lambda^{d+\infty/2}$ defined by Uglov ([20], 4.1). This space has a basis which Uglov indexes by $\mathcal{P}$ or by pairs (λ, s) where $\lambda \in \mathcal{P}^m$ and $s = (s_p)$ is an m-tuple of integers with $\sum_p = s_p = d$. There is a bijection between the two index sets given by $\lambda \to (\lambda^*, s)$ where λ^* is the m-quotient of λ and s is a particular labeling of the m-core of λ ([20], 4.1, 4.2).

There is a subspace $\mathcal{F}^{(s)}_{m,\ell}$ of $\mathcal{F}^{(d)}_{m,\ell}$, the Fock space associated with (s), which is a weight space for the $\widehat{\mathfrak{sl}_\ell}$ action ([19], p.982). We have $\mathcal{F}^{(d)}_{m,\ell} = \oplus \mathcal{F}^{(s)}_{m,\ell}$, the sum of weight spaces. Here we can define a basis $\{|\lambda, s>\}$ with $\lambda \in \mathcal{P}^\ell$ of ${\mathcal{F}_{m,\ell}}^{(s)}$. The spaces $\mathcal{F}^{(s)}_{m,\ell}$ were also studied by Uglov.

The endomorphism of $\mathbb{C}^m \otimes \mathbb{C}[t, t^{-1}]$ induced by multiplication by t^r gives rise to a linear operator b_r and its adjoint b'_r on ${\mathcal{F}_m}^{(d)}$, and thus to an action of $\mathfrak{H}$ on ${\mathcal{F}_m}^{(d)}$. We also have an action of $\mathfrak{H}$ by operators b_r, b'_r on ${\mathcal{F}_{m,\ell}}^{(s)}$, and this is the main result that we need ([19], p.982).

We now choose a fixed ℓ-tuple s. With suitable parameters of $H(\Gamma_n)$, each n, the $\mathbb{C}$-vector space $[\mathcal{O}(\Gamma)]$ is then canonically isomorphic to $\mathcal{F}^{(s)}_{m,\ell}$. We then have the following $\mathbb{C}$-linear isomorphisms ([19], p.990, 5.20):

$$\begin{gathered}
[\mathcal{O}(\Gamma)] \to R(\Gamma) \to \mathcal{F}^{(s)}_{m,\ell}, \\
\Delta_\lambda \to \bar{L}_\lambda \to |\lambda, s> .
\end{gathered} \tag{5.1}$$

Consider the Fock space $\mathcal{F}^{(s)}_{m,\ell}$ with basis indexed by $\{|\lambda, s>\}$ where $\lambda \in \mathcal{P}^\ell$. The element $b_{s_\mu} \in \mathfrak{H}$, i.e. b_f where $f = s_\mu$, a Schur function, acts by an operator S_μ on the space. The functor a^*_μ on $[\mathcal{O}(\Gamma)]$ (see Section 3) is now identified with S_μ by ([19], Proposition 5.13, p.990).

Remark. The bijection between m-core partitions and the m-tuples (s) as above has been studied by combinatorialists (see e.g. [11]).

6. Fock space revisited

References for the combinatorial definitions in this section are [14], [15]. Given a partition μ we introduce three operators on Fock space: an operator S_μ defined by Leclerc and Thibon [14], an operator $\mathcal{F}^*(\phi_\mu)$ defined by Farahat [7] on representations of the symmetric groups $\mathcal{S}_n$, and the operators $\mathcal{L}_\mu$ of Lusztig induction on G_n. The algebra of symmetric functions in $\{x_1, x_2, \ldots\}$ is denoted by Λ.

The integers ℓ, m in Section 5 will now be replaced by a positive integer e which was used in the context of blocks of G_n. Thus $\Gamma_n = \mu_e \wr \mathcal{S}_n$.

First consider the space $\mathcal{F}^{(d)}_e$ where $d \in \mathbb{Z}$, with basis elements $\{|\lambda, d>\}$ where $\lambda \in \mathcal{P}$. Leclerc and Thibon [14] introduced elements in $U(\mathfrak{H})$ which we write in our previous notation as b_{h_ρ} and b_{s_μ}, acting as operators V_ρ and

S_μ on $\mathcal{F}_e^{(d)}$ where $\rho, \mu \in \mathcal{P}$ and h_ρ is a homogeneous symmetric function. These operators have a combinatorial description as follows. Here we will write $|\lambda >$ for $|\lambda, d >$.

First they define commuting operators V_k, ($k \geq 1$ on $\mathcal{F}_e^{(d)}$ defined by

$$V_k(|\lambda >) = \sum_\mu (-1)^{-s(\mu/\lambda)} |\mu >, \tag{6.1}$$

where the sum is over all μ such that μ/λ is a horizontal n-ribbon strip of weight k, and $s(\mu/\lambda)$ is the "spin" of the strip.

Here a ribbon is the same as a rim-hook, i.e. a skew- partition which does not contain a 2×2 square. The head of the ribbon is the upper right box and the tail is the lower left box. The spin is the leg length of the ribbon, i.e. the number of rows -1.

Definition 6.1. *(see* [12]*) A horizontal n-ribbon strip of weight k is a tiling of a skew partition by k n-ribbons such that the topright-most square of every ribbon touches the northern edge of the shape. The spin of the strip is the sum of the spins of all the ribbons.*

It can be shown that a tiling of a skew partition as above is unique. More generally we can then define V_ρ where ρ is a composition. If $\rho = \{\rho_1, \rho_2, \ldots\}$ then $V_\rho = V_{\rho_1}.V_{\rho_2}\ldots$. Finally we define operators S_μ acting on $\mathcal{F}_e^{(d)}$ which we connect to Lusztig induction.

Definition 6.2. $S_\mu = \sum_\rho \kappa_{\mu\rho} V_\rho$ *where the $\kappa_{\mu\rho}$ are inverse Kostka numbers (*[14]*, p.204), (*[12]*, p.8).*

Remark. Let $p_e(f)$ denote the plethysm by the power function in Λ, i.e. $p_e(f(x_1, x_2, \ldots)) = f(x_1^e, x_2^e, \ldots)$. (This is related to a Frobenius morphism; see [15], p.171.) In fact in [14] $\mathfrak{H}$ is regarded as a $\mathbb{C}(q)$-space where q is an indeterminate. Then V_ρ and S_μ are q-analogs of multiplication by $p_e(h_\rho)$ and $p_e(s_\mu)$ in Λ.

Next, let $\mathcal{A}_n$ be the category of unipotent representations of G_n. Let $\mathcal{A} = \oplus_{n\geq 0}[\mathcal{A}_n]$. We recall from Section 4 that the unipotent characters of G_n are denoted by $\{\chi_\lambda\}$ where $\lambda \vdash n$. We now regard $\mathcal{A}$ as having a basis $[\chi_\lambda]$ where λ runs through all partitions. Then $\mathcal{A}$ is isomorphic to $\mathcal{F}_e^{(d)}$ as a $\mathbb{C}$-vector space, since $\mathcal{A}$ also has a basis indexed by partitions.

We now define Lusztig operators $\mathcal{L}_\mu$ on $\mathcal{A}$ and then relate them to the S_μ.

Definition 6.3. *Let $\mu \vdash k$. The Lusztig map $\mathcal{L}_\mu : \mathcal{A} \to \mathcal{A}$ is as follows. Define $\mathcal{L}_\mu : [\mathcal{A}_n] \to [\mathcal{A}_{n+ke}]$ by $[\chi_\lambda] \to [R_L^{G_{n+ke}}(\chi_\lambda \times \chi_\mu)]$, where $L = G_n \times GL(k, q^e)$ or $L = G_n \times U(k, q^e)$, an e-split Levi subgroup of G_{n+ke}.*

Finally, consider the characters of $\mathcal{S}_n$. We denote the character corresponding to $\lambda \in \mathcal{P}_n$ as ϕ_λ. We also use $\lambda \in \mathcal{P}_n$ to denote representatives of conjugacy classes of $\mathcal{S}_n$. Let $\mathcal{C}_n$ be the category of representations of $\mathcal{S}_n$ and $\mathcal{C} = \oplus_{n \geq 0}[\mathcal{C}_n]$.

Given partitions $\nu \vdash (n+ke)$, $\lambda \vdash n$ such that ν/λ is defined, Farahat [7] has defined a character $\hat{\phi}_{\nu/\lambda}$ of S_k, as follows. Let the e-tuples $(\nu^{(i)})$, $(\lambda^{(i)})$ be the e-quotients of ν and λ. Then $\epsilon \prod_i \phi_{(\nu^{(i)}/\lambda^{(i)})}$, where $\epsilon = \pm 1$ is a character of a Young subgroup of S_k, which induces up to the character $\hat{\phi}_{\nu/\lambda}$ of S_k.

We will instead use an approach of Enguehard ([6], p.37) which is more conceptual and convenient for our purpose.

Definition 6.4. *Let $\mu \vdash k$. The Farahat map $\mathcal{F} : [\mathcal{C}_{ek}] \to [\mathcal{C}_k]$ is defined by $(\mathcal{F}\chi)(\mu) = \chi(e\mu)$. It is then extended to the map $\mathcal{F} : [\mathcal{C}_n] \to [\mathcal{C}_k] \times [\mathcal{C}_\ell]$, by first restricting a representation of S_n to $S_{ek} \times S_\ell$ and then applying $\mathcal{F}$ to $[\mathcal{C}_{ek}]$ and the identity to $[\mathcal{C}_\ell]$.*

Fix $\mu \vdash k$. Taking adjoints and denoting $\mathcal{F}^*$ by $\mathcal{F}^*(\phi_\mu)$ we then have, for $\lambda \vdash n$:

Definition 6.5. $\mathcal{F}^*(\phi_\mu) : [\mathcal{C}_n] \to [\mathcal{C}_{n+ek}], \quad \phi_\lambda \to \mathrm{Ind}_{\mathcal{S}_{ek} \times \mathcal{S}_n}^{\mathcal{S}_{n+ek}} (\mathcal{F}(\phi_\mu) \times \phi_\lambda)$.

By the standard classification of maximal tori in G_n we can denote a set of representatives of the G_n-conjugacy classes of the tori by $\{T_w\}$, where w runs over a set of representatives for the conjugacy classes of S_n. We then have that the unipotent character $\chi_\lambda = \frac{1}{|S_n|} \sum_{w \in S_n} \lambda(w) R_{T_w}^{G_n}(1)$ (see e.g. ([8], 1.13). Here, as before, $R_{T_w}^{G_n}(1)$ is Lusztig induction.

We assume in the proposition below that when $G_n = U(n,q)$ that $e \equiv 0 \pmod 4$. This is the case that is analogous to the case of $GL(n,q)$. The other cases for e require some straightforward modifications which we mention below. The proof of the proposition has been sketched by Enguehard ([6],p.37) when $G_n = GL(n,q)$.

Let M be the e-split Levi subgroup of G_n isomorphic to $GL(k,q^e) \times GL_\ell$. We denote by ${}^*R_M^G$ the adjoint of the Lusztig map R_M^G. It is an analogue of the map $\mathcal{F}^*$, and this is made precise below.

Given $\lambda \vdash n$, we have a bijection $\phi_\lambda \leftrightarrow \chi_\lambda$ between $[\mathcal{C}_n]$ and $[\mathcal{A}_n]$. We then have an obvious bijection $\psi : \phi_\lambda \leftrightarrow \chi_\lambda$ between $\mathcal{C}$ and $\mathcal{A}$.

Proposition 6.1. *Let $G = GL(ek,q)$ or $U(ek,q)$. In the case of $U(ek,q)$ we assume $e \equiv 0 \pmod 4$. Let $M \cong GL(k,q^e)$, a subgroup of G. Let $\psi : \phi_\lambda \leftrightarrow \chi_\lambda$ between $\mathcal{C}$ and $\mathcal{A}$ be as above.*

Then (i) If $\lambda \vdash ek$, $\psi(\mathcal{F}(\phi_\lambda)) =^* R_M^G(\chi_\lambda)$

(ii) If $\mu \vdash k$, $\psi(\mathcal{F}^*(\phi_\mu)) = R_M^G(\chi_\mu) = \mathcal{L}_\mu$.

Proof. We have $\psi(\mathcal{F}(\phi_\lambda)) = \frac{1}{|\mathcal{S}_k|}\sum_{w\in\mathcal{S}_k}(\mathcal{F}\phi_\lambda)(w)R^M_{T_w}(1) = \frac{1}{|\mathcal{S}_k|}\sum_{w\in\mathcal{S}_k}\phi_\lambda(ew)R^M_{T_w}(1)$. Since the torus parametrized by w in M is parametrized by ew in G, we can write this as $\frac{1}{|\mathcal{S}_k|}\sum_{w\in\mathcal{S}_k}\phi_\lambda(ew)R^M_{T_{ew}}(1)$.

On the other hand, we have (see [8], Lemma 2B), using the parametrization of tori in M, ${}^*R^G_M(\chi_\lambda) = \frac{1}{|\mathcal{S}_k|}\sum_{w\in\mathcal{S}_k}\phi_\lambda(w)R^M_{T_w}(1)$. This proves (i). Then (ii) follows by taking adjoints. ☐

The proposition clearly generalizes to the subgroup $M \cong GL(k.q^e)\times G_\ell$ of G_n where $n = ek+\ell$. In the case of $U(n.q)$, if e is odd we replace e by e' where $e' = 2e$ with $M \cong GL(k, q^{e'})$, and if $e \equiv 2 \bmod 4$ by e' where $e' = e/2$ with $M \cong U(k, q^{e'})$, the proof being similar.

Using the isomorphism between the spaces $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{F}_e^{(d)}$, we now regard the operators $\mathcal{L}_\mu$, $\mathcal{F}^*(\phi_\mu)$ and $\mathcal{S}_\mu$ as acting on $\mathcal{F}_e^{(d)}$.

We now prove one of the main results in this paper.

Theorem 6.1. *The operators $\mathcal{L}_\mu$ and S_μ on $\mathcal{F}_e^{(d)}$ coincide.*

Proof. We have shown above that $\mathcal{F}^*(\phi_\mu) = \mathcal{L}_\mu$. We will now show that $\mathcal{F}^*(\phi_\mu) = S_\mu$.

More generally we consider the character $\hat{\phi}_{\nu/\lambda}$ of S_k defined by Farahat, where $\nu \vdash (n+ke)$ and $\mu \vdash n$, and describe it using $\mathcal{F}$. The restriction of ϕ_ν to $S_n \times S_{ke}$ can be written as a sum of $\phi_\lambda \times \phi_{\nu/\lambda}$ where $\phi_{\nu/\lambda}$ is a (reducible) character of S_{ke}, and characters not involving ϕ_λ. We then define $\hat{\phi}_{\nu/\lambda} = \mathcal{F}(\phi_{\nu/\lambda})$, a character of S_k. We then note that $\hat{\phi}_{\nu/\lambda}(u) = \phi_{\nu/\lambda}(eu)$. Using the characteristic map we get a corresponding skew symmetric function s_{ν^*/λ^*}. This is precisely the function which has been described in ([16], p.91), since it is derived from the usual symmetric function $s_{\nu/\lambda}$ by taking e-th roots of variables. Using the plethysm function p_e and its adjoint ψ_e ([13],p.1048)) there we get $s_{\nu^*/\lambda^*} = \psi_e(s_{\nu/\lambda})$.

By the above facts we get

$$\begin{aligned}&(\hat{\phi}_{\nu/\lambda}, \phi_\mu)\\ &= (s_{\nu^*/\lambda^*}, s_\mu)\\ &= (\psi_e(s_{\nu/\lambda}), s_\mu)\\ &= (s_{\nu/\lambda}, p_e(s_\mu)),\\ &= (p_e(s_\mu).s_\lambda, s_\nu)\\ &= (S_\mu[\chi_\lambda], [\chi_\nu]).\end{aligned}$$

The last equality can be seen as follows. There is a $\mathbb{C}$-linear isomorphism between the algebra Λ and $\mathcal{F}_e^{(d)}$, since both have bases indexed by $\mathcal{P}$. Under this isomorphism multiplication by the symmetric function $p_e(s_\mu)$ on Λ corresponds to the operator S_μ on Fock space (see [14], p.6).

This proves that $\mathcal{L}_\mu = S_\mu$. $\square$

We recall that $\widehat{sl}_e$ acts on $\mathcal{F}_e^{(d)}$ and hence on $\mathcal{A}$.

Corollary 6.1. *The highest weight vectors $V_\rho\emptyset$ of the irreducible components of the $\widehat{sl}_e$-module $\mathcal{A}$ ([13],p.1054) can be described by Lusztig induction.*

Remark. In fact Leclerc and Thibon also have a parameter q in their definition of S_μ, since they deal with a deformed Fock space. Thus S_μ can be regarded as a quantized version of a Lusztig operator $\mathcal{L}_\mu$.

Remark. In the notation of ([15], p.173) we have $(s_{\nu^*/\lambda^*},s_\mu)=(s_{\nu_0/\lambda_0}s_{\nu_1/\lambda_1}\ldots s_{\nu_{e-1}/\lambda_{e-1}},s_\mu = c^\mu_{\nu/\lambda}$, where the $c^\mu_{\nu/\lambda}$ are Littlewood-Richardson coefficients. We now have $(\chi_\nu, R_M^{G_n}(\chi_\lambda \times \chi_\mu)) = \epsilon c^\mu_{\nu/\lambda}$, where $\epsilon = \pm 1$. In particular $c^{(k)}_{\nu/\lambda}$ is the number of tableaux of shape ν such that ν/λ is a horizontal e-ribbon of weight k. Thus the Lusztig operator $\mathcal{L}_k$ can be described in terms of e-ribbons of weight k, similar to the case of $k = 1$ which classically is described by e-hooks.

7. CRDAHA and Lusztig induction

The main reference for parabolic induction in this section is [19].

In this section we show a connection between the parabolic induction functor a_μ^* on $[\mathcal{O}(\Gamma)]$ and the Lusztig induction functor $\mathcal{L}_\mu$ in $\mathcal{A}$ using Fock space. In particular this gives an explanation of the Global to Local Bijection for G_n given in Theorem 4.2. This can be regarded as a local, block-theoretic version of Theorem 6.1.

As mentioned in Section 4, the unipotent characters χ_λ in an e-block of G_n are constituents of the Lusztig map $R_L^{G_n}(\lambda)$ where (L, λ) is an e-cuspidal pair. Up to sign, they are in bijection with the characters of $W_{G_n}(L, \lambda)$, and they all have the same e-core.

For our result we can assume $d = 0$, which we do from now on. We set $\ell = m = e$ as in Section 6. We have spaces $\mathcal{F}_e^{(0)}$ and $\mathcal{F}_{e,e}^{(0)} = \oplus_s \mathcal{F}_{e,e}^{(s)}$ where $s = (s_p)$ is an e-tuple of integers with $\sum_p s_p = 0$. We now fix such an s.

By ([19], 6.17, 6.22, p.1010) we have a $U(\mathfrak{H})$-isomorphism between $\mathcal{F}_e^{(0)}$ and $\mathcal{F}_{e,e}^{(0)}$. Let $\mathcal{F}_e^{(s)}$ be the inverse image of $\mathcal{F}_{e,e}^{(s)}$ under this isomorphism. We then have $\mathbb{C}$-isomorphisms from $\mathcal{F}_{e,e}^{(s)}$ to $[\mathcal{O}(\Gamma)]$, and from $\mathcal{F}_e^{(s)}$ to $\mathcal{A}^{(s)}$, where $\mathcal{A}^{(s)}$ is the the subspace of $\mathcal{A}$ spanned by $[\chi_\lambda]$ where the χ_λ are in an e-block parametrized by the e-core labeled by (s) (see Section 5).

The spaces $\mathcal{F}_e^{(s)}$, $\mathcal{F}_{e,e}^{(s)}$, $[\mathcal{O}(\Gamma)]$, $\mathcal{A}^{(s)}$ have bases $\{|\lambda, s >: \lambda \in \mathcal{P}\}$, $\{|\lambda, s >: \lambda \in \mathcal{P}^e\}$, $\{\Delta_\lambda : \lambda \in \mathcal{P}^e\}$ and $[\chi_\lambda]$ where λ has e-core labeled by s, respectively.

We have maps $S_\mu : \mathcal{F}_{e,e}^{(s)} \to \mathcal{F}_{e,e}^{(s)}$, $\mu \in \mathcal{P}^e$, $S_\mu : \mathcal{F}_e^{(s)} \to \mathcal{F}_e^{(s)}$, $\mu \in \mathcal{P}$, $\mathcal{L}_\mu : \mathcal{A}^{(s)} \to \mathcal{A}^{(s)}$ and $a_\mu^* : [\mathcal{O}(\Gamma)] \to [\mathcal{O}(\Gamma)]$.

The following theorem can be regarded as a refined version of the Global to Local Bijection of [1]. The case $e = 1$ is due to Enguehard ([6], p.37), where the proof is a direct verification of the theorem from the definition of the Farahat map $\mathcal{F}$ in $\mathcal{S}_n$ (see Section 6) and Lusztig induction in G_n.

Theorem 7.1. *Under the isomorphism $\mathcal{A}^{(s)} \cong [\mathcal{O}(\Gamma)]$ given by $[\chi_\lambda] \to [\Delta_{\lambda^*}]$ where λ^* is the e-quotient of λ, Lusztig induction $\mathcal{L}_\mu$ on $\mathcal{A}^{(s)}$ with $\mu \in \mathcal{P}$ corresponds to parabolic induction a_μ^* on $[\mathcal{O}(\Gamma)]$ with $\mu \in \mathcal{P}^e$.*

Proof. Consider the action of $b_{s_\mu} \in U(\mathfrak{H})$ on $\mathcal{F}_{e,e}^{(s)}$. The operator S_μ acting on $\mathcal{F}_{e,e}^{(s)}$ can be identified with a_μ^* acting on $[\mathcal{O}(\Gamma)]$, with the basis element $|\lambda, s>$ corresponding to $[\Delta_\lambda]$ ([19], 5.20).

On the other hand, $b_{s_\mu} \in U(\mathfrak{H})$ acts as S_μ on the space $\mathcal{F}_e^{(s)}$ and thus, by Theorem 6.1 as $\mathcal{L}_\mu$ on $\mathcal{A}^{(s)}$ with the basis element $|\lambda, s>$ corresponding to $[\chi_\lambda]$. Here we note that Lusztig induction preserves e-cores, and thus $\mathcal{L}_\mu$ fixes $\mathcal{A}^{(s)}$.

Now $\mathcal{F}_e^{(s)}$ is isomorphic to $\mathcal{A}^{(s)}$ and $\mathcal{F}_{e,e}^{(s)}$ is isomorphic to $[\mathcal{O}(\Gamma)]$. Thus we have shown that a_μ^* and $\mathcal{L}_\mu$ correspond under two equivalent representations of $U(\mathfrak{H})$.

□

Corollary 7.1. *The BMM-bijection of Theorem 4.2 between the constituents of the Lusztig map $R_L^{G_n}(\lambda)$ where (L, λ) is an e-cuspidal pair and the characters of $W_{G_n}(L, \lambda)$ is described via equivalent representations of $U(\mathfrak{H})$ on Fock spaces.*

This follows from the theorem, using the map spe (see Section3).

8. Decomposition numbers

References for this section are [5], [14], [15]. In this section we assume $G_n = GL(n, q)$, since we will be using the connection with q-Schur algebras. We describe connections between weight spaces of $\widehat{\mathfrak{sl}_e}$ on Fock space, blocks of q-Schur algebras, and blocks of G_n. We show that some Brauer characters of G_n can be described by Lusztig induction.

The ℓ-decomposition numbers of the groups G_n have been studied by Dipper-James and by Geck, Gruber, Hiss and Malle. The latter have also studied the classical groups, using modular Harish-Chandra induction. One of the key ideas in these papers is to compare the decomposition matrices of the groups with those of q-Schur algebras.

We have the Dipper-James theory over a field of characteristic 0 or ℓ. They define (**??**, 2.9) the q-Schur algebra $\mathcal{S}_q(n)$, endomorphism algebra of a sum of permutation representations of the Hecke algebra $\mathcal{H}_n$ of type A_{n-1} [5]. The unipotent characters and the ℓ-modular Brauer characters of G_n are both indexed by partitions of n (see [8]). Similarly the Weyl modules and the simple modules of $\mathcal{S}_q(n)$ are both indexed by partitions of n (see [5]).

For $\mathcal{S}_q(n)$ over k of characteristic ℓ, $q \in k$, one can define the decomposition matrix of $\mathcal{S}_q(n)$, where q is an e-th root of unity, where as before e is the order of q mod ℓ. By the above this is a square matrix whose entries are the multiplicities of simple modules in Weyl modules. Dipper-James ([5], 4.9) showed that this matrix, up to reordering the rows and columns, is the same as the unipotent part of the ℓ-decomposition matrix of G_n, the transition matrix between the ordinary (complex) characters and the ℓ-modular Brauer characters. The rows and columns of the matrices are indexed by partitions of n.

We consider the Fock space $\mathcal{F} = \mathcal{F}_e{}^{(d)}$ for a fixed d, which as in Section 6 is isomorphic to $\mathcal{A}$, and has the standard basis $\{|\lambda >\}, \lambda \in \mathcal{P}$. It also has two canonical bases $G^+(\lambda)$ and $G^-(\lambda)$, $\lambda \in \mathcal{P}$ ([14],[15]). There is a recursive algorithm to determine these two bases.

We fix an s as in Section 6). The algebra $\widehat{\mathfrak{sl}_e}$ acts on $\mathcal{F}_{e,e}^{(s)}$ and hence on $\mathcal{F}_e^{(s)}$, which is a weight space for the algebra. The connection between $\widehat{\mathfrak{sl}_e}$-weight spaces and blocks of the q-Schur algebras and hence blocks of $GL(n,q), n \geq 0$ is known, and we describe it below. We denote the Weyl module of $\mathcal{S}_q(n)$ parametrized by λ by $W(\lambda)$.

We need to introduce a function res on $\mathcal{P}$. If $\lambda \in \mathcal{P}$, The e-residue of the (i,j)-node of the Young diagram of λ is the non-negative integer r given by $r \equiv j - i (\text{mod } e)$, $0 \leq r < e$, denoted $\text{res}_{i,j}(\lambda)$. Then $\text{res}(\lambda) = \cup_{(i,j)}(\text{res}_{i,j}(\lambda))$.

Proposition 8.1. *A weight space for $\widehat{\mathfrak{sl}_e}$ on $\mathcal{F}_e^{(s)}$ can be regarded as a block of a q-Schur algebra with q a primitive e-th root of unity.*

Proof. Two Weyl modules $W(\lambda), W(\mu)$ are in the same block if and only if $\text{res}(\lambda) = \text{res}(\mu)$ (see e.g. ([17], Theorem 5.5, $(i) \Leftrightarrow (iv)$). The fact that *res* defines a weight space follows e.g. from ([18], p.60). ☐

Thus a weight space determines a set of partitions of a fixed $n \geq 0$.

Corollary 8.1. *A weight space for $\widehat{\mathfrak{sl}_e}$ on $\mathcal{F}_e^{(s)}$ can be regarded as a block of $GL(n,q)$, where n is determined from the weight space.*

We now have the following theorem which connects the ℓ-decomposition numbers of $G_n, n \geq 0$ with Fock space.

Theorem 8.1. *Let ϕ_μ be the Brauer character of G_n indexed by $\mu \in \mathcal{P}_n$. Let $\lambda \in \mathcal{P}_n$.Then, for large ℓ, $(\chi_\mu, \phi_\lambda) = (G^-(\lambda), |\mu >)$, where $(G^-(\lambda), |\mu >)$ is the coefficient of $|\mu >)$ in the expansion of the canonical basis in terms of the standard basis.*

Proof. The decomposition matrix of $\mathcal{S}_q(n)$ over a field of characteristic 0, with q a root of unity, is known by Varagnolo-Vasserot [21]. By their work the coefficients in the expansion of the $G^+(\lambda)$ in terms of the standard basis give the decomposition numbers for the algebras $\mathcal{S}_q(n)$, $n \geq 0$, with q specialized at an e-th root of unity (see [**?**]).

By an asymptotic argument of Geck [10] we can pass from the decomposition matrices of q-Schur algebras in characteristic 0 to those in characteristic ℓ, where ℓ is large. Then by the Dipper-James theorem we can pass to the decomposition matrices of the groups G_n over a field of characteristic ℓ with q an e-th root of unity in the field.

Let D_n be the unipotent part of the ℓ-decomposition matrix of G_n and E_n its inverse transpose. Thus D_n has columns $G^+(\lambda)$ and E_n has rows given by $G^-(\lambda)$ (see [14], Section 4). The rows of E_n also give the Brauer characters of G_n, in terms of unipotent characters. These two descriptions of the rows of E_n then gives the result. ☐

The following analog of Steinberg's Tensor Product Theorem is proved for the canonical basis $G^-(\lambda)$ in [14].

Theorem 8.2. *Let λ be a partition such that λ' is e-singular, so that $\lambda = \mu + e\alpha$ where μ' is e-regular. Then $G^-(\lambda) = S_\alpha G^-(\mu)$.*

We now show that the rows indexed by partitions λ as in the above theorem can be described by Lusztig induction. By replacing S_α by $\mathcal{L}_\alpha$ and using Theorem 6.1 it follows that in these cases, Lusztig-induced characters coincide with Brauer characters.

Theorem 8.3. *Let $\lambda = \mu + e\alpha$ where μ' is e-regular. Then the Brauer character represented by $G^-(\lambda)$ is equal to the Lusztig generalized character $R_L^{G_n}(G^-(\mu) \times \chi_\alpha)$, where $L = G_m \times GL(k, q^e)$, $n = m + ke, \alpha \vdash k$.*

By using the BMM bijection, Theorem 4.2, we have the following corollary.

Corollary 8.2. *Let $\mu = \phi$, so that $\lambda = e\alpha$. Then the Brauer character represented by $G^-(\lambda)$ can be calculated from an induced character in a complex reflection group.*

Examples, thanks to GAP [9]:

Some tables giving the basis vectors $G^-(\lambda), e = 2$ are given in [15]. In our examples we use transpose partitions of the partitions in these tables, and rows instead of columns.

We first give an example of a weight space for $\widehat{sl}_e$, which is also a block for G_n, with $n = 4$, $e = 4$. This is an example of a decomposition matrix D for $n = 4$, $e = 4$:

$$\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 \\ 31|| & 1 & 1 & 0 & 0 \\ 211|| & 0 & 1 & 1 & 0 \\ 1111|| & 0 & 0 & 1 & 1 \end{pmatrix}$$

The following example is to illustrate Theorem 8.3.

An example of the inverse of a decomposition matrix for $n = 6$, $e = 2$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix}$$

Here the rows are indexed as: $6, 51, 42, 41^2, 3^2, 31^3, 2^3, 2^21^2, 21^4, 1^6$

In the above matrix:

The rows indexed by $1^6, 2^21^2, 3^2, 21^4, 41^2$ have interpretations as Brauer characters, in terms of $R_L^{G_n}$, with L e-split Levi of the form $GL(3, q^2)$ for $\lambda = 1^6, 2^21^2, 3^2$, of the form $GL(2, q) \times GL(2, q^2)$ for $\lambda = 21^4$, and of the form $GL(4, q) \times GL(1, q^2)$ for $\lambda = 41^2$.

With $L = GL(3, q^2)$,

Row indexed by 3^2 is $R_L^G(\chi_3) = \chi_{3^2} - \chi_{42} + \chi_{51} - \chi_6$,

Row indexed by 2^21^2 is $R_L^G(\chi_{21})$ and

Row indexed by 1^6 is $R_L^G(\chi_{1^3})$.

9. Acknowledgement

The author thanks Bernard Leclerc for valuable discussions on this paper.

References

[1] M. Broué, G. Malle & J. Michel, Représentations Unipotentes génériques et blocs des groupes réductifs finis, Astérisque 212 (1993).

[2] R.Bezrukavnikov and P.Etingof, Parabolic induction and restriction functors for rational Cherednik algebras, Selecta math. 14 (2009), 397-425.

[3] M. Cabanes and M. Enguehard, Representation theory of finite reductive groups, New Mathematical Monographs, 1, Cambridge University Press, Cambridge, 2004.

[4] J.Chuang & K.M.Tan, Canonical basis of the basic $U_q(\widehat{sl_n})$-module, J.Algebra 248 (2002), 765-779.

[5] R.Dipper and G.James, The q-Schur algebra, *Proc. London Math. Soc.* **59** (1989), 23-50.

[6] M. Enguehard. Combinatoire des Symboles et Application de Lusztig. *Publicationes du LMENS* **92-9** (1992).

[7] H.Farahat, On the representations of the symmetric group, Proc. London Math. Soc. 4 (1954), 303-316.

[8] P.Fong and B.Srinivasan, The blocks of finite general linear and unitary groups, *Invent. Math.* **69** (1982), 109–153.

[9] The GAP Group: GAPGroups, Algorithms, and Programming, Version 4.4.10. http://www.gap-system.org (2007)

[10] M.Geck, Modular Harish-Chandra series, Hecke algebras and (generalized) q-Schur algebras. *In*: Modular Representation Theory of Finite Groups (Charlottesville, VA, 1998; eds. M. J. Collins, B. J. Parshall and L. L. Scott), p. 1–66, Walter de Gruyter, Berlin 2001

[11] F. Garvan, D. Kim and D.Stanton, Cranks and t-cores, *Invent. Math.* **1-01** (1990), 1-17.

[12] T.Lam, Ribbon Tableaux and the Heisenberg algebra, *Math. Zeit.* **250** (2005), 685-710.

[13] A.Lascoux, B.Leclerc, J-Y. Thibon, Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras and unipotent varieties, *J.Math.Phys.* **38** (1997), 447-456.

[14] B.Leclerc, J-Y. Thibon, Canonical bases of q-deformed Fock spaces, *Int. math. Res. Notices* **9** (1996), 447-456.

[15] B.Leclerc, J-Y. Thibon, Littlewood-Richardson coefficients and Kazhdan-Lusztig polynomials, Advanced Studies in Pure Math. 28 (2000), 155-220.

[16] I.G.Macdonald, *Symmetric functions and Hall polynomials*, Oxford (1995).

[17] A.Mathas, *The representation Theory of the Ariki-Koike and cyclotomic q-Schur algebras*, Adv. Stud. Pure Math. **40,** 261-320, Math. Soc. Japan, Tokyo, 2004.

[18] R.Rouquier, P.Shan, M.Varagnolo, E.Vasserot, Categorifications and cyclotomic rational double affine Hecke algebras, arXiv: 1305.4456v1

[19] P.Shan, E.Vasserot, Heisenberg algebras and rational double affine Hecke algebras, *J. Amer. Math. Soc.* **25** (2012), 959-1031.

[20] D. Uglov, Canonical bases of higher level q-deformed Fock spaces and Kazhdan-Lusztig poly- nomials, *Progress in Math.* **191**, Birkhauser (2000).

[21] M.Varagnolo, E.Vasserot, On the decomposition numbers of the quantized Schur algebra, Duke Math. J. 100 (1999), 267-297.

Department of Mathematics, Statistics, and Computer Science (MC 249), University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607-7045

E-mail address: `srinivas@uic.edu`