Combinatorics in Representations of Finite Classical Groups

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Combinatorics plays a role in the Representation Theory (Ordinary and Modular) of:

(i) Symmetric Groups
(ii) General linear groups (over $\mathbb{F}_q$ and $\mathbb{C}$)
(iii) Classical groups (over $\mathbb{F}_q$ and $\mathbb{C}$)
Some main problems of modular representation theory:

- Describe the irreducible modular representations, e.g. their degrees
- Describe the blocks
- Find the decomposition matrix $D$, the transition matrix between ordinary and Brauer characters.
- Global to local: Describe information on the block $B$ by ”local information”, i.e. from blocks of subgroups of the form $N_G(P)$, $P$ a $p$-group
Ordinary characters of $S_n$: parametrized by partitions of $n$

Given partition $\lambda$, have $\chi_\lambda \in \text{Irr}(S_n)$

$\lambda$ has associated Young tableau

Hook length formula:

$$\chi_\lambda(1) = \frac{n!}{\prod h_{ij}}$$

Here $h_{ij}$ is the hook length of node $(i, j)$
Given $\lambda$, have $\chi_\lambda \in \text{Irr}(S_n)$

- $p$ positive integer: Have $p$-hooks, $p$-core of $\lambda$.

Theorem (Brauer-Nakayama) Characters $\chi_\lambda$, $\chi_\mu$ of $S_n$ are in the same $p$-block ($p$ prime) if and only if $\lambda$ and $\mu$ have the same $p$-core.
Recently, concept of $p$-weight of $\lambda = \text{number of } p\text{-hooks removed to get to the } p\text{-core}.$

Theorem (Chuang-Rouquier, 2005) Two $p$-blocks of $S_n$ with the same $p$-weight are derived equivalent, i.e. the derived categories of the block algebras are equivalent.
$G$ connected reductive group over $F_q$, $F = \bar{F}_q$

$q$ a power $p^n$ of the prime $p$

$F$ Frobenius endomorphism, $F : G \rightarrow G$

$G = G^F$ finite reductive group

$T$ torus, closed subgroup $\simeq F^\times \times F^\times \times \cdots \times F^\times$

$L$ Levi subgroup, centralizer $C_G(T)$ of a torus $T$
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Let $\mathbf{P}$ be an $F$-stable parabolic subgroup of $\mathbf{G}$ and $\mathbf{L}$ an $F$-stable Levi subgroup of $\mathbf{P}$ so that $L \leq P \leq G$.

Harish-Chandra induction is the following map:

$$R_L^G : K_0(KL) \rightarrow K_0(KG).$$

If $\psi \in \text{Irr}(L)$ then $R_L^G(\psi) = \text{Ind}_P^G(\tilde{\psi})$ where $\tilde{\psi}$ is the character of $P$ obtained by inflating $\psi$ to $P$. 
Let $P$ be an $F$-stable parabolic subgroup of $G$ and $L$ an $F$-stable Levi subgroup of $P$ so that $L \subseteq P \subseteq G$.

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\( \chi \in \text{Irr}(G) \) is **cuspidal** if \( \langle \chi, R_L^G(\psi) \rangle = 0 \) for any \( L \leq P < G \) where \( P \) is a proper parabolic subgroup of \( G \). The pair \((L, \theta)\) a cuspidal pair if \( \theta \in \text{Irr}(L) \) is cuspidal.

\( \text{Irr}(G) \) partitioned into Harish-Chandra families: A family is the set of constituents of \( R_L^G(\theta) \) where \((L, \theta)\) is cuspidal.
Now let $\ell$ be a prime not dividing $q$.
Suppose $L$ is an $F$-stable Levi subgroup, not necessarily in an $F$-stable parabolic $P$ of $G$.

- The Deligne-Lusztig linear operator:

$$R^G_L : K_0(\overline{Q}L) \rightarrow K_0(\overline{Q}G).$$

- Every $\chi$ in $\text{Irr}(G)$ is in $R^G_T(\theta)$ for some $(T, \theta)$, where $T$ is an $F$-stable maximal torus and $\theta \in \text{Irr}(T)$.

- The unipotent characters of $G$ are the irreducible characters $\chi$ in $R^G_T(1)$ as $T$ runs over $F$-stable maximal tori of $G$.

If $L \leq P \leq G$, where $P$ is a $F$-stable parabolic subgroup, $R^G_L$ is just Harish-Chandra induction.
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If \( L \leq P \leq G \), where \( P \) is a \( F \)-stable parabolic subgroup, \( R^G_L \) is just Harish-Chandra induction.
Example: $G = GL(n, q)$. If $L$ is the subgroup of diagonal matrices contained in the (Borel) subgroup of upper triangular matrices, we can do Harish-Chandra induction. But if $L$ is a torus (Coxeter torus) of order $q^n - 1$, we must do Deligne-Lusztig induction to obtain generalized characters from characters of $L$. 
As before, $G$ is a finite reductive group. If $e$ is a positive integer, $\phi_e(q)$ is the $e$-th cyclotomic polynomial. The order of $G$ is the product of a power of $q$ and certain cyclotomic polynomials. A torus $T$ of $G$ is a $\phi_e$-torus if $T$ has order a power of $\phi_e(q)$.

The centralizer in $G$ of a $\phi_e$-torus is an $e$-split Levi subgroup of $G$. 
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Example. In $GL_n$, $e$-split Levi subgroups $L$ are isomorphic to

$$\prod_i GL(m_i, q^e) \times GL(r, q).$$

An $e$-cuspidal pair $(L, \theta)$ is defined as in the Harish-Chandra case, using only $e$-split Levi subgroups. Thus $\chi \in \text{Irr}(G)$ is $e$-cuspidal if

$$\langle \chi, R^G_L(\psi) \rangle = 0$$

for any $e$-split Levi subgroup $L$.

The unipotent characters of $G$ are divided into $e$-Harish-Chandra families, as in the usual Harish-Chandra case of $e = 1$. 
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The unipotent characters of $G$ are divided into $e$-Harish-Chandra families, as in the usual Harish-Chandra case of $e = 1$. 
Definition. A unipotent block of $G$ is a block which contains unipotent characters.

SURPRISE: Brauer Theory and Lusztig Theory are compatible!

THEOREM (Cabanes-Enguehard) Let $B$ be a unipotent block of $G$, $\ell$ odd and good, $e$ the order of $q \mod \ell$. Then the unipotent characters in $B$ are precisely the constituents of $R^G_L(\lambda)$ where the pair $(L, \lambda)$ is $e$-cuspidal.

Thus the unipotent blocks of $G$ are parametrized by $e$-cuspidal pairs $(L, \lambda)$ up to $G$-conjugacy.
$G = GL(n, q)$, $\ell$ a prime not dividing $q$, $e$ the order of $q \mod \ell$.

The unipotent characters of $G$ are indexed by partitions of $n$. Degrees again by a hook length formula:

$$\chi_{\lambda}(1) = |G|/\prod(q^{h_{ij}-1})$$

Theorem (Fong-Srinivasan, 1982) $\chi_{\lambda}, \chi_{\mu}$ are in the same $\ell$-block if and only if $\lambda, \mu$ have the same $e$-core.
Interpretation: \( \chi_\lambda, \chi_\mu \) are in the same \( \ell \)-block if and only if they are constituents of \( R_L^G(\psi) \) where \( L \) is a product of tori of order \( q^{e-1} \) and \( GL(m, q), \psi = 1 \times \chi_\kappa, \kappa \) is an \( e \)-core.

\( \lambda, \mu \) are obtained from \( \kappa \) by adding \( e \)-hooks. Blocks are classified by \( e \)-cores.
Interpretation: $\chi_\lambda, \chi_\mu$ are in the same $\ell$-block if and only if they are constituents of $R^G_L(\psi)$ where $L$ is a product of tori of order $q^{e-1}$ and $GL(m, q), \psi = 1 \times \chi_\kappa, \kappa$ is an $e$-core.

$\lambda, \mu$ are obtained from $\kappa$ by adding $e$-hooks. Blocks are classified by $e$-cores.
$G = \text{Sp}(2n, q), \text{SO}(2n + 1, q), \text{SO}^\pm(2n, q), \text{q odd.}$

e is the order of $q$ mod $\ell$.

A symbol is a pair $(\Lambda_1, \Lambda_2)$ of subsets of $\mathbb{N}$. Notion of $e$-hooks, $e$-cohooks, $e$-cores of symbols defined.

\[
\begin{pmatrix}
0 & 1 & 2 \\
1 & 3
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 4 \\
1 & 3
\end{pmatrix},
\begin{pmatrix}
1 & 3 & 4 \\
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\end{pmatrix}
\]

Get the second and third symbols from the first by adding a 2-hook, 2-cohook.
$G = Sp(2n, q), SO(2n + 1, q), SO^\pm(2n, q)$, $q$ odd.

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Get the second and third symbols from the first by adding a 2-hook, 2-cohook.
In $G = \text{Sp}(2n, q), \text{SO}(2n + 1, q), \text{SO}^{\pm}(2n, q)$, unipotent blocks are again classified by $e$-cores of symbols. (Fong-Srinivasan, 1989)

Theorem of Asai gives the $R^G_L(\lambda)$ map where $\lambda$ is parametrized by symbols: Add $e$-hooks or $e$-cohooks.
Aim: Bijection of non-unipotent blocks with unipotent blocks of suitable subgroups, e.g. centralizers of semisimple elements.

Theorem of Bonnafé- Rouquier: Case when centralizers are Levi.

Theorem on unipotent blocks of $G = Sp(2n, q), SO(2n + 1, q), SO^\pm(2n, q)$ generalized to "quadratic unipotent" blocks. Centralizers of semisimple elements are products of symplectic, orthogonal groups. Blocks of $O(2n + 1, q), O^\pm(2n, q)$ involved; pairs of symbols, e-hooks, e-cohooks arise.
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Let $G$ be “twisted” $GL(n, q)$, i.e. $GL(n, q)$ extended by an automorphism of order 2. Have blocks of $G$ parametrized by pairs of partitions, connected with blocks of subgroups of the form $Sp(2r, q) \times O(n - 2r, q)$.

Recall H. Weyl: $GL_n$ is the ”all-embracing majesty”!
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