QUADRATIC UNIPOTENT BLOCKS IN GENERAL LINEAR, UNITARY AND SYMPLECTIC GROUPS

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Abstract. An irreducible ordinary character of a finite reductive group is called quadratic unipotent if it corresponds under Jordan decomposition to a semisimple element s in a dual group such that $s^2 = 1$. We prove that there is a bijection between, on the one hand the set of quadratic unipotent characters of $GL(n, q)$ or $U(n, q)$ for all $n \geq 0$ and on the other hand, the set of quadratic unipotent characters of $Sp(2n, q)$ for all $n \geq 0$. We then extend this correspondence to ℓ-blocks for certain ℓ not dividing q.

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1. Introduction

Let $\mathbb{G}$ be a connected, reductive algebraic group defined over $\mathbb{F}_q$ and G the finite reductive group of $\mathbb{F}_q$-rational points of $\mathbb{G}$. The irreducible characters of G are divided into rational Lusztig series $\mathcal{E}(G, (s))$ where (s) is a semisimple conjugacy class in a dual group G^* of G. Let ℓ be a prime not dividing q. Each ℓ-block of G also determines a conjugacy class (s) in G^*, where now s is an ℓ'-semisimple element. The block is said to be isolated if $C_{\mathbb{G}^*}(s)$ is not contained in a proper Levi subgroup of $\mathbb{G}^*$. If a block is not isolated, the characters in the block in $\mathcal{E}(G, (s))$ can be obtained by Lusztig induction from a Levi subgroup of G. Thus it is important to study the isolated blocks of G. A description of the characters in isolated blocks of classical groups when ℓ and q are odd and q is large was given in [19] and [20].

On the other hand, the notion of a perfect isometry between blocks with abelian defect groups of two finite groups was introduced by M.Broué [2]. This leads to a comparison between an ℓ-block B of a finite group G and an ℓ-block b of a group H. If there is a perfect isometry between B and b, certain invariants of the blocks are preserved. Often H is a "local subgroup" of G, for example the normalizer of a defect group of B. In other situations G and H are finite groups of the same type, e.g. symmetric groups , general linear groups or unitary groups. (In fact there is a stronger result, i.e. the abelian defect group conjecture, for symmetric groups and general linear groups; see [6].)

In this paper we study quadratic unipotent characters, i.e. characters in Lusztig series with $s^2 = 1$, and quadratic unipotent blocks, i.e. blocks which contain quadratic unipotent characters, of general linear, unitary and symplectic groups. Here we assume that q and ℓ are odd. These blocks include unipotent blocks and are isolated blocks for the symplectic group. We first show that there is a natural bijection between the quadratic unipotent characters of $GL(n, q)$ or $U(n, q)$ and the quadratic unipotent characters of a suitable symplectic group $Sp(2m, q)$. Let e be the order of q mod ℓ. If B is a quadratic unipotent block of $GL(n, q)$ with e even or of $U(n, q)$ with e odd or $e \equiv 0(\mathrm{mod} 4)$ we show that there is a perfect isometry between B and a quadratic unipotent block b of a symplectic group $Sp(2m, q)$. This kind of connection between groups of type A and C appears to be new.

Our main tool is the combinatorics of partitions and symbols related to the blocks of general linear and symplectic groups. In particular our work is inspired by a paper of Waldspurger [22]; a map which is defined there between two combinatorial configurations can be used to set up correspondences between blocks as above.

The paper is organized as follows. In Section 2 we describe the construction and parametrization of quadratic unipotent characters in $GL(n, q)$, $U(n, q)$ and $Sp(2n, q)$. Our main theorem, Theorem 2.1, gives a bijection between the sets of quadratic unipotent characters in $GL(n, q)$ or $U(n, q)$ for all $n \geq 0$ and the corresponding sets in $Sp(2n, q)$ for all $n \geq 0$. In Section 3 we parametrize quadratic unipotent blocks with e as above for these groups, and in Section 4 we prove correspondences between blocks of $GL(n, q)$ or $U(n, q)$ for all $n \geq 0$ and blocks of $Sp(2n, q)$ for all $n \geq 0$. In Section 5 we construct perfect isometries between corresponding blocks, in the case of abelian defect groups. Finally in Section 6 we give an alternative interpretation of the above correspondences. For the groups $G = GL(n, q)$ or $G = U(n, q)$ and $H = Sp(2n, q)$, we consider groups $G(s)$ and $H(s)$ constructed by Enguehard as dual groups to the centralizers of a semisimple element s with $s^2 = 1$ in groups dual to G or H. We then interpret our correspondences as between unipotent blocks of $G(s)$ and $H(s)$.

Notation: If G is a finite group, $\mathrm{Irr}(G)$ is the set of (complex) irreducible characters of G. The Weyl group of type B_n is denoted by W_n. The Grothendieck group of an abelian category $\mathcal{C}$ is denoted by $K_0(\mathcal{C})$.

2. Quadratic Unipotent Characters

If G is a finite reductive group the set $\mathrm{Irr}(G)$ is partitioned into geometric series by Deligne-Lusztig theory, and further into rational series $\mathcal{E}(G, (s))$ where $s \in G^*$ is a semisimple element (see [4], 8.23). For the groups G that we study we assume throughout this paper that q is odd and ℓ is an odd prime not dividing q.

Definition 2.1. *If $\chi \in \mathcal{E}(G,(s))$ where s satisfies $s^2 = 1$ we say χ is a quadratic unipotent character.*

These characters were called square-unipotent in [20]. In particular we have the unipotent characters, where $s = 1$. If $G = Sp(2n, q)$ (resp. $SO^{\pm}(2n, q)$) then $G^* = SO(2n + 1, q)$ (resp. $SO^{\pm}(2n, q)$), and if $G = GL(n, q)$ or $G = U(n, q)$ then $G = G^*$. Since q is odd, if $s^2 = 1$ where $s \in G^*$ we get quadratic unipotent characters in $\mathcal{E}(G,(s))$.

Let $G_n = GL(n, q)$ or $U(n, q)$. The unipotent characters of G_n are parametrized by partitions of n. More generally, quadratic unipotent characters of $GL(n, q)$ have been explicitly constructed by Waldspurger ([22]). We generalize his construction also to $U(n, q)$ below.

Let (μ_1, μ_2) be a pair of partitions where μ_i is a partition of n_i, $i = 1, 2$, with $n_1 + n_2 = n$. Let $L = G_{n_1} \times G_{n_2}$ be a Levi subgroup of G_n, where G_{n_i} is a general linear or a unitary group according as $G_n = GL(n, q)$ or $U(n, q)$. Let $\mathcal{E}$ be the unique linear character of G_{n_2} of order 2 and let χ_{μ_i} be the unipotent character of G_{n_i} corresponding to the partition μ_i. Then the virtual character $R_L^{G_n}(\chi_{\mu_1} \times \mathcal{E}\chi_{\mu_2})$ obtained by Lusztig induction from L (which in fact is Harish-Chandra induction when $G_n = GL(n, q)$) is a quadratic unipotent character, up to sign. We denote it by $\chi_{(\mu_1,\mu_2)}$. All quadratic unipotent characters of G_n are obtained this way, and thus we have a parametrization of quadratic unipotent characters by pairs (μ_1, μ_2) such that $|\mu_1| + |\mu_2| = n$. (We note also that by abuse of notation we use the finite groups when we write $R_L^{G_n}$.)

An alternative description of the quadratic unipotent characters of $G_n = GL(n, q)$ or $U(n, q)$ is given as follows. These characters are precisely the constituents of $R_L^{G_n}(1 \times \mathcal{E} \times \chi_{(\kappa_1,\kappa_2)})$, where L is a Levi subgroup of the form $T_1 \times T_2 \times G_{n_0}$, T_1 (resp. T_2) is a product of N_1 (resp. N_2) tori of order $q^2 - 1$, and 1 (resp. $\mathcal{E}$) is the trivial character (resp. character of order 2) of T_1 (resp. T_2). The character $\chi_{(\kappa_1,\kappa_2)}$ is a 2-cuspidal character of G_{n_0}, i.e. κ_1 and κ_2 are 2-cores. We note that in this case, by the work of Lusztig ([13]) the $R_L^{G_n}$ map is Harish-Chandra induction for $U(n, q)$. The endomorphism algebra of the induced representation is isomorphic to a Hecke algebra of type $W_{N_1} \times W_{N_2}$.

Let $H_n = Sp(2n, q)$, q odd. We have a similar description of quadratic unipotent characters of H_n, as given by Lusztig ([13]) and Waldspurger

([21], 4.9). The characters are constituents of $R_K^{H_n}(1 \times \mathcal{E} \times \chi)$, where K is a Levi subgroup of the form $T_1 \times T_2 \times H_{n_0}$, T_1 (resp. T_2) is a product of N_1 (resp. N_2) tori of order $q-1$, and 1 (resp. $\mathcal{E}$) is the trivial character (resp. character of order 2) of T_1 (resp. T_2). The character χ is a cuspidal quadratic unipotent character of H_{n_0} and the $R_K^{H_n}$ map is Harish-Chandra induction . The endomorphism algebra of the induced representation is again isomorphic to a Hecke algebra of type $W_{N_1} \times W_{N_2}$.

We now describe the combinatorics of symbols needed to parameterize the quadratic unipotent characters of H_n. By the work of Lusztig [13] the unipotent characters of classical groups are parameterized by equivalence classes of symbols. We refer to ([3]), ([1], p. 48) for a description of the symbols associated with unipotent characters of $Sp(2n, q)$, including definitions of the equivalence relations on symbols and the rank and defect of a symbol.

We denote a symbol by $\Lambda = (S, T)$ where $S, T \subseteq \mathbb{N}$. If Λ is unordered, it is regarded as the same as (T, S) and also the same as the symbol obtained by a shift operation from itself (see [3], p. 375). The defect of Λ is $|S| - |T|$. We also need to consider ordered symbols to parameterize unipotent characters of $O^{\pm}(2n, q)$, which were described by Waldspurger.

We then have:

- The unipotent characters of $Sp(2n, q)$ are in bijection with unordered symbols of rank n and odd defect.
- The unipotent characters of $O^+(n, q)$ are in bijection with ordered symbols of rank n and defect $\equiv 0 \pmod 4$
- The unipotent characters of $O^-(n, q)$ are in bijection with ordered symbols of rank n and defect $\equiv 2 \pmod 4$
- The irreducible characters of W_n are in bijection with unordered symbols of rank n and defect 1.

The operations of "adding an a-hook" to and "deleting an a-hook" from a partition, and the concept of an "a-core" of a partition are well-known. Similarly we have operations of "adding an a-hook or an a-cohook" and "deleting an a-hook or a-cohook" to a symbol Λ. They can be described as follows (see [17], p.-226). Let $\Lambda = (S, T)$. We say a symbol Λ' is obtained from Λ by adding an a-hook if it is obtained by deleting a member x of S (or T) and inserting $x + a$ in S (or T). We say Λ' is obtained from Λ by adding an a-cohook if it is obtained from Λ by deleting a member x of S (or T) and inserting $x + a$ in T (or S).

We follow the notation of [21] below. We define a map σ on ordered symbols by $\sigma(S, T) = (T, S)$. Let $\widetilde{S}_{n,d}$ be the set of ordered symbols of rank n and defect d, and let $S_{n,d} = \widetilde{S}_{n,d} \cup \widetilde{S}_{n,-d}$, modulo the relation $\Lambda \sim \sigma(\Lambda)$.

Let

$$\begin{aligned} S_{n,odd} &= \bigcup_{\substack{d\in\mathbb{N}\\ d \text{ odd}}} S_{n,d},\ \widetilde{S}_{n,\,even} = \bigcup_{\substack{d\in\mathbb{Z}\\ d \text{ even}}} \widetilde{S}_{n,d}, \\ S\widetilde{S}_{n,\text{ mix}} &= \bigcup_{n_1+n_2=n} (S_{n_1,\text{ odd}} \times \widetilde{S}_{n_2,\text{ even}}). \end{aligned}$$

Remark. We have taken the liberty of replacing “pair” by “even” and “imp” by “odd” in [21].

By the work of Lusztig [13] and Waldspurger [21] we have a parametrization of the quadratic unipotent characters of H_n by $S\widetilde{S}_{n,\text{ mix}}$ which generalizes that of the unipotent characters, given above. This will be clarified in Lemma 2.2 below.

We note that if $\rho \in W_n$ there is a symbol of defect 1 corresponding to ρ (see [3], p.375). By abuse of notation we will sometimes refer to ”the core (or cocore) of ρ”, to mean the core (or cocore) of the symbol.

We now give the parametrization of the quadratic unipotent characters of G_n and H_n which we will use in our description of blocks. We remark that the parametrization by 4-tuples in the case of G_n, rather than by pairs of partitions is crucial for our results.

Lemma 2.1. *The quadratic unipotent characters of G_n can be parameterized by 4-tuples $(m_1, m_2, \rho_1, \rho_2)$ such that*

$$m_1(m_1+1)/2 + m_2(m_2+1)/2 + 2N_1 + 2N_2 = n,$$

where $m_1, m_2 \in \mathbb{N}$ and $\rho_i \in \mathrm{Irr}\ W_{N_i}$, $i = 1, 2$. If $\chi_{(\mu_1,\mu_2)}$ is parameterized by $(m_1, m_2, \rho_1, \rho_2)$ then the 2-core of μ_i is $\{m_i, m_{i-1}, \dots, 2, 1\}$ and the 2-quotient of μ_i is $\rho_i \in \mathrm{Irr}\ W_{N_i}$, $i = 1, 2$.

Proof. The quadratic unipotent characters of G_n are parameterized by pairs of partitions (μ_1, μ_2) such that $|\mu_1| + |\mu_2| = n$. A combinatorial proof that we may parameterize these characters of G_n by 4-tuples $(m_1, m_2, \rho_1, \rho_2)$ as above is given in ([22], p.361).

We note that the character parameterized by $(m_1, m_2, -, -)$ is 2-cuspidal for $GL(n, q)$. In the case of $U(n, q)$ the description given above also shows that we can regard this parametrization as coming from Harish-Chandra induction from a suitable Levi subgroup L, with $(m_1, m_2, -, -)$ the parameters for a cuspidal quadratic unipotent character of a possibly smaller unitary group $U(n_0, q)$ and with (ρ_1, ρ_2) the character of a product of two Hecke algebras of type B corresponding to $W_{N_1} \times W_{N_2}$. This gives another proof of the parametrization by the 4-tuples as above for $U(n, q)$, and hence for $GL(n, q)$. ∎

Remark. For an explanation of the connection between the two parametrizations of unipotent characters of $U(n,q)$ see also ([12], p.224).

Lemma 2.2. *The quadratic unipotent characters of H_n can be parameterized by pairs of symbols (Λ_1, Λ_2) and by 4-tuples $(h_1, h_2, \rho_1, \rho_2)$ such that $h_1(h_1+1) + h_2^2 + N_1 + N_2 = n$, where $h_1 \in \mathbb{N}$, $h_2 \in \mathbf{Z}$ and $\rho_i \in \mathrm{Irr}\ W_{N_i}$, $i = 1, 2$*

Proof. As in the case of $U(n,q)$ this is done by Harish-Chandra induction of cuspidal quadratic-unipotent characters from a suitable Levi subgroup K (see Waldspurger [21], 4.9-4.11). The endomorphism algebra of the induced representation is again isomorphic to a Hecke algebra of type $W_{N_1} \times W_{N_2}$. Hence the set of quadratic unipotent characters of H_n is parametrized by 4-tuples $(h_1, h_2, \rho_1, \rho_2)$, where the cuspidal character is parameterized by $(h_1, h_2, -, -)$. Then ([21], 2.21, 4.10) the pair (h_1, ρ_1) corresponds to a symbol $\Lambda_1 \in S_{h_1+h_1^2+N_1,odd}$ and the pair (h_2, ρ_2) corresponds to a symbol $\Lambda_2 \in \widetilde{S}_{h_2^2+N_2,even}$. Thus there is a pair $(\Lambda_1, \Lambda_2) \in S\widetilde{S}_{n,mix}$ corresponding to the 4-tuple $(h_1, h_2, \rho_1, \rho_2)$, and there is a bijection of $S\widetilde{S}_{n,mix}$ with the set of quadratic unipotent characters of H_n. We note that if $(\Lambda_1, \Lambda_2) \in S\widetilde{S}_{n,mix}$ the corresponding character is in $\mathcal{E}(H_n, (s))$ where the number of eigenvalues of s equal to 1 (resp. -1) in the natural representation of the dual group $SO(2n+1)$ is 2 $\mathrm{rank}(\Lambda_1) + 1$ (resp. 2 $\mathrm{rank}(\Lambda_2)$).

We also note here the connection between the symbols Λ_1, Λ_2 and the symbols corresponding to ρ_1, ρ_2. Suppose the symbol corresponding to ρ_1 is (S, T) where $|S| = |T| + 1$. Then the symbol corresponding to Λ_1 is (S', T) where, if $2h_1 + 1 = d$, $S' = \{[0, d-2] \cup (S + d - 1)\}$ ([13],3.2). The formula for ρ_2 and Λ_2 is similar. □

The quadratic unipotent character parameterized by (Λ_1, Λ_2) is denoted by $\chi_{(\Lambda_1,\Lambda_2)}$. Since $(\Lambda_1, \Lambda_2) \in S\widetilde{S}_{n,mix}$, Λ_1 is an unordered symbol and Λ_2 is an ordered symbol.

The following lemma is a first step towards connecting the quadratic unipotent characters of the groups G_n and the groups H_n.

Lemma 2.3. *(see [22], p.362). There is a bijection between pairs (m_1, m_2) such that $m_1(m_1+1)/2 + m_2(m_2+1)/2 = n$ and pairs (h_1, h_2) such that $h_1(h_1+1) + h_2^2 = n$. This bijection is defined by $m_1 = sup(h_1 + h_2, -h_1 - h_2 - 1)$ and $m_2 = sup(h_1 - h_2, h_2 - h_1 - 1)$.*

Remark. Note that if h_2 is replaced by $-h_2$, m_1 and m_2 are interchanged in the above bijection.

This bijection then leads to the following result, which is crucial to us. The proof is a straightforward extension of the above lemma.

Theorem 2.1. *The map* $(m_1, m_2, \rho_1, \rho_2) \to (h_1, h_2, \rho_1, \rho_2)$, $\rho_i \in \mathrm{Irr}\ W_{N_i}$, $i = 1, 2$ *is a bijection between the set of quadratic unipotent characters of* G_n, *and the set of quadratic unipotent characters of* H_m, *where* $m_1(m_1 + 1)/2 + m_2(m_2 + 1)/2 + 2N_1 + 2N_2 = n$ *and* $h_1(h_1 + 1) + h_2^2 + N_1 + N_2 = m$. *Under this bijection the character corresponding to* $(m_1, m_2, -, -)$ *maps to the character corresponding to* $(h_1, h_2, -, -)$.

Example. The group $Sp(4, q)$ has 23 quadratic unipotent characters (and only 6 unipotent characters). Of these, 14 characters are in bijection with quadratic unipotent characters of $GL(4, q)$, 8 with those of $GL(3, q)$ and 1 with that of $GL(2, q)$. The latter is the unipotent cuspidal character θ_{10}, which is in bijection with the quadratic unipotent (not unipotent) 2-cuspidal character of $GL(2, q)$ parameterized by the pair of partitions $(1, 1)$ or by the 4-tuple $(1, 1, -, -)$. Here $m_1 = m_2 = 1, h_1 = 1, h_2 = N_1 = N_2 = 0$.

Example. The group $GL(4, q)$ has 20 quadratic unipotent characters (and only 5 unipotent characters). Of these, 14 characters are in bijection with quadratic unipotent characters of $Sp(4, q)$, 4 with those of $Sp(6, q)$ and 2 with those of $Sp(8, q)$. The latter are cuspidal quadratic unipotent characters of $Sp(8, q)$ corresponding to cuspidal quadratic unipotent characters of $O^+(8, q)$ under Jordan decomposition. They are in bijection with the quadratic unipotent 2-cuspidal characters of $GL(4, q)$ parameterized by the pair of partitions $(21, 1)$. Here $m_1 = 2, m_2 = 1, h_2 = 2, h_1 = N_1 = N_2 = 0$, or $(1, 21)$ with $m_1 = 1, m_2 = 2, h_2 = -2, h_1 = N_1 = N_2 = 0$.

Theorem 2.1 can be restated as follows. Let L_n (resp. L'_n) be the category of quadratic unipotent characters of G_n (resp. H_n).

Theorem 2.2. *There is an isomorphism (isometry) between the groups* $\oplus_{n\geq 0}\ K_0(L_n)$ *and* $\oplus_{n\geq 0}\ K_0(L'_n)$ *given by mapping the character parameterized by* $(m_1, m_2, \rho_1, \rho_2)$ *to the character parameterized by* $(h_1, h_2, \rho_1, \rho_2)$, $\rho_i \in \mathrm{Irr}\ W_{N_i}$, $i = 1, 2$.

3. Quadratic unipotent blocks

The ℓ - blocks of G_n and of the conformal symplectic group $CSp(2n, q)$ were classified in [10], [11]. We define a quadratic unipotent block of G_n or H_n to be one which contains quadratic unipotent characters. As a special case we have the unipotent blocks, which have been studied by many authors (see e.g. [4]). The quadratic unipotent ℓ - blocks of H_n for $q > 2n$ were classified in [19]. A description of the characters in a quadratic unipotent block of H_n was given in [20] if $q > 2n$.

The following theorem describes these results. Here and in the rest of the paper, e is the order of q mod ℓ and f the order of q^2 mod ℓ.

Theorem 3.1. *(i)* [10] *Let ℓ divide q^f+1 if $G_n = GL(n,q)$ and let ℓ divide q^f+1, f even, or q^f-1, f odd, if $G_n = U(n,q)$. Let B be a quadratic unipotent ℓ-block of G_n. Then B corresponds to a pair (λ_1, λ_2) of partitions such that $|\lambda_1| + |\lambda_2| = n'$ and such that λ_1 and λ_2 are e-cores, i.e. no $2f$-hooks can be removed from them. The quadratic unipotent characters in B are of the form $\chi_{(\mu_1,\mu_2)}$ where λ_i is the e-core of μ_i $(i = 1, 2)$. These characters are precisely the constituents of $R_L^{G_n}(1 \times \mathcal{E} \times \chi_{(\lambda_1,\lambda_2)})$, where L is a Levi subgroup of the form $T_1 \times T_2 \times G_{n'}$, T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order $q^{2f}-1$, and 1 (resp. $\mathcal{E}$) is the trivial character (resp. character of order 2) of T_1 (resp. T_2). The character $\chi_{(\lambda_1,\lambda_2)})$ is in a block of defect 0 of $G_{n'}$.*

(ii) [20] *Let $q > 2m$. Let b be a quadratic unipotent ℓ - block, i.e. an isolated block of H_m and let ℓ divide q^f-1, f odd. Then b corresponds to a pair of symbols (π_1, π_2) where the π_i are f-cores . The quadratic unipotent characters in b are of the form $\chi_{(\Lambda_1,\Lambda_2)}$ where π_i is the f-core of Λ_i $(i = 1, 2)$. These characters are precisely the constituents of $R_K^{H_m}(1\times\mathcal{E}\times\chi_{(\pi_1,\pi_2)})$, where K is a Levi subgroup of the form $T_1 \times T_2 \times H_{m'}$, T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order q^f-1 and 1 (resp. $\mathcal{E}$) is the trivial character (resp. character of order 2) of T_1 (resp. T_2). The character $\chi_{(\pi_1,\pi_2)}$ is in a block of defect 0 of $H_{m'}$.*

(iii) [20] *Let $q > 2m$. Let b be a quadratic unipotent ℓ - block, i.e. an isolated block of H_m and let ℓ divide q^f+1. Then b corresponds to a pair of symbols (π_1, π_2) where the π_i are f-cocores. The quadratic unipotent characters in b are of the form $\chi_{(\Lambda_1,\Lambda_2)}$ where π_i is the f-cocore of Λ_i $(i = 1, 2)$. These characters are precisely the constituents of $R_K^{H_m}(1 \times \mathcal{E} \times \chi_{(\pi_1,\pi_2)})$, where K is a Levi subgroup of the form $T_1 \times T_2 \times H_{m'}$, T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order q^f+1 and 1 (resp. $\mathcal{E}$) is the trivial character (resp. character of order 2) of T_1 (resp. T_2). The character $\chi_{(\pi_1,\pi_2)}$ is in a block of defect 0 of $H_{m'}$.* □

The following combinatorial lemma due to Olsson ([17], p.235) and to Enguehard ([9],5.7) will be used to connect blocks of types (ii) and (iii) in the above theorem.

Lemma 3.1. *Given a symbol Λ of rank n and a positive integer e one can define a symbol $\widehat{\Lambda}$, called the e-twisting of Λ in ([17],p.235) such that there is a bijection between e-cohooks in Λ and e-hooks in $\widehat{\Lambda}$. In particular if $\widehat{\Lambda}$ is an e-core, i.e. has no e-hooks, then Λ is an e-cocore, i.e. has no e-cohooks.*

Corollary 3.1. *The operation of e-twisting is an involution on the set of quadratic unipotent characters of $Sp(2n, q)$.*

Theorem 3.2. *If $G_n = GL(n,q)$, let $e = 2f$ be the order of q mod ℓ, so that ℓ divides $q^f + 1$. (We exclude the case where e is odd.) If $G_n = U(n,q)$ let again e be the order of q mod ℓ and f the order of q^2 mod ℓ. Consider the two cases: (i) $e = f$ is odd, ℓ divides $q^{2f} - 1$ and $q^f - 1$, or (ii) $e = 2f$ where f is even, i.e. $e \equiv 0 \pmod 4$ and ℓ divides $q^f + 1$. The case $e \equiv 2 \pmod 4$ is excluded. Then the quadratic unipotent blocks of G_n are parameterized by 6-tuples $(m_1, m_2, \sigma_1, \sigma_2, M_1, M_2)$, where $\sigma_i \in \mathrm{Irr}\ W_{N'_i}$, $i = 1, 2$ with $fM_1 + N'_1 = N_1$, $fM_2 + N'_2 = N_2$, $m_1(m_1+1)/2 + m_2(m_2+1)/2 + 2N_1 + 2N_2 = n$. The quadratic unipotent characters in a block parameterized by $(m_1, m_2, \sigma_1, \sigma_2, M_1, M_2)$ are then parameterized by 4-tuples $(m_1, m_2, \rho_1, \rho_2)$ such that (ρ_1, ρ_2) have (σ_1, σ_2) as f-cores.*

Proof. We use Theorem 3.1 and the construction of quadratic unipotent characters. Let B be a quadratic unipotent ℓ-block of G_n. We have the following configurations, by our choice of e. The block B corresponds to an e-split Levi subgroup of G_n which is a product of $M_1 + M_2$ tori of order $q^{2f} - 1$ and $G_{n'}$. Then $G_{n'}$ has a 2-split Levi subgroup which is a product of $N_1{}' + N_2{}'$ tori of order $q^2 - 1$ and G_{n_0}, and finally G_n has a 2-split Levi subgroup which is a product of $N_1 + N_2$ tori of order $q^2 - 1$ and G_{n_0}.

Then B corresponds to a pair (λ_1, λ_2) of $2f$-cores which parameterize a block of defect 0 of $G_{n'}$. Suppose the 2-core of (λ_1, λ_2) is (κ_1, κ_2). Then (κ_1, κ_2) is parameterized by a 4-tuple $(m_1, m_2, -, -)$, where κ_i is the partition $(m_i, m_i - 1, \ldots 1)$ for $i = 1, 2$. Then the $2f$-core (λ_1, λ_2) is parameterized by a 4-tuple $(m_1, m_2, \sigma_1, \sigma_2)$, where $\sigma_i \in \mathrm{Irr}\ W_{N'_i}$, $i = 1, 2$, and $m_1(m_1+1)/2 + m_2(m_2+1)/2 + 2N'_1 + 2N'_2 = n'$. Since B is parameterized by the pair of the e-split Levi subgroup and the character (λ_1, λ_2), we get the parametrization of B by the sextuple $(m_1, m_2, \sigma_1, \sigma_2, M_1, M_2)$, where $\sigma_i \in \mathrm{Irr}\ W_{N'_i}$, $i = 1, 2$.

Let $\chi_{(\mu_1, \mu_2)} \in B$. Now λ_1 and λ_2 are obtained from μ_1 and μ_2 respectively by removing $2f$-hooks. Removing a $2f$-hook can be achieved by removing f 2-hooks. Thus all the (μ_1, μ_2) parameterizing the quadratic unipotent characters in B have the same 2-core (κ_1, κ_2). Then all the 4-tuples parameterizing the quadratic unipotent characters in B have the form $(m_1, m_2, \rho_1, \rho_2)$ such that $m_1(m_1+1)/2 + m_2(m_2+1)/2 + 2N_1 + 2N_2 = n$, where $\rho_i \in \mathrm{Irr} W_{N_i}$, $i = 1, 2$. In other words the pair (m_1, m_2) is fixed for all the characters. We then note (see Lemma 2.1) that (σ_1, σ_2) are the 2-quotients of the partitions (λ_1, λ_2), and hence σ_1 and σ_2 are f-cores. A count of the number of 2-hooks removed from a pair of partitions to reach the 2-core gives

$$fM_1 + N'_1 = N_1, \ fM_2 + N'_2 = N_2.$$

This gives the result. $\square$

The proof of the next proposition for the groups H_m and the case of ℓ dividing $q^f - 1$ is similar to the above.

Theorem 3.3. *Let $q > 2m$. Let ℓ divide $q^f - 1$, f odd. The quadratic unipotent blocks of H_m are parameterized by 6-tuples $(h_1, h_2, \sigma_1, \sigma_2, M_1, M_2)$, where $\sigma_i \in \mathrm{Irr}\ W_{N_i'}$, $i = 1, 2$ with $fM_1 + N_1' = N_1$, $fM_2 + N_2' = N_2$, $h_1(h_1+1) + h_2^2 + N_1 + N_2 = m$. Here the symbols corresponding to σ_1 and σ_2 are f-cores. The quadratic unipotent characters in b are parameterized by 4-tuples of the form $(h_1, h_2, \rho_1, \rho_2)$ where (ρ_1, ρ_2) have (σ_1, σ_2) as f-cores.*

Proof. Let b be a quadratic unipotent ℓ-block of H_m corresponding to a pair of symbols (π_1, π_2) which are f-cores, as in Theorem 3.1. The 1-core of (π_1, π_2) is parameterized by $(h_1, h_2, -, -)$ for some h_1, h_2 and (π_1, π_2) is parameterized by $(h_1, h_2, \sigma_1, \sigma_2)$, where $\sigma_i \in \mathrm{Irr}\ W_{N_i'}$, $i = 1, 2$. To show that σ_1 is an f-core, we can assume $\pi_1 = (S', T)$, and that the symbol corresponding to σ_1 is (S, T) as in Lemma 2.2. Using the description given there of the connection between S and S' it is easy to see that removing an f-hook from (S', T) is equivalent to removing an f-hook from (S, T). Thus (S', T) is an f-core if and only if (S, T) is an f-core.

Let $\chi_{(\Lambda_1, \Lambda_2)} \in b$. Now π_1 and π_2 are obtained from Λ_1 and Λ_2 respectively by removing f-hooks . Removing an f-hook can be achieved by removing f 1-hooks. Thus all the (Λ_1, Λ_2) parameterizing the quadratic unipotent characters in b have the same 1-core which is the 1-core of (π_1, π_2).

Furthermore all the 4-tuples parameterizing the quadratic unipotent characters in b have the form $(h_1, h_2, \rho_1, \rho_2)$ such that $h_1(h_1+1)+h_2^2+N_1+N_2 = m$, where $\rho_i \in \mathrm{Irr}\ W_{N_i}$, $i = 1, 2$. In other words the pair (h_1, h_2) is fixed for all the characters. As before we have $fM_1 + N_1' = N_1$, $fM_2 + N_2' = N_2$ where M_1, M_2 are as in Theorem 3.1 (ii). If $\chi_{(\Lambda_1, \Lambda_2)}$ is parameterized by $(h_1, h_2, \rho_1, \rho_2)$ then the above arguments on removing f-hooks applied to (Λ_1, Λ_2) and the symbols corresponding to (ρ_1, ρ_2) show that since (Λ_1, Λ_2) have (π_1, π_2) as f-cores, (ρ_1, ρ_2) have (σ_1, σ_2) as f-cores. □

Remark . The above arguments show that the pair (ρ_1, ρ_2) can be regarded as the 1-quotient of the pair (Λ_1, Λ_2). This is a special case of the concept of an e-quotient of a symbol in ([17], Lemma 9).

The case of H_m where ℓ divides q^f+1 will be considered after proving Lemma 4.2 below, since in that case we have to use cohooks instead of hooks.

Remark . The 4-tuple $(m_1, m_2, \sigma_1, \sigma_2)$ (resp. $(h_1, h_2, \sigma_1, \sigma_2)$) can be regarded as the "core" of the block B (resp. b), and the pair (M_1, M_2) can be regarded as the "weight" of the block.

4. Correspondences between blocks

The parametrization of blocks described in the last section leads to the main theorems of this section. When considering a block of H_m, it will be assumed that $q > 2m$. The block correspondences that we derive in Theorems 4.1,4.2 will be between blocks of G_n where $m_1(m_1+1)/2 + m_2(m_2+1)/2 + 2N_1 + 2N_2 = n$ and blocks of H_m where $h_1(h_1+1) + h_2^2 + N_1 + N_2 = m$.

Theorem 4.1. *Let $\ell|(q^f - 1)$, f odd. Let B be a quadratic unipotent block of $U(n,q)$ parameterized by $(m_1, m_2, \sigma_1, \sigma_2, M_1, M_2)$, where σ_1 and σ_2 are f-cores. Let (m_1, m_2) correspond under Waldspurger's map to the pair (h_1, h_2), and let b be the block of $Sp(2m, q)$ parameterized by $(h_1, h_2, \sigma_1, \sigma_2, M_1, M_2)$. Then B and b correspond in the sense that (i) their defect groups are isomorphic, and (ii) there is a natural bijection between the quadratic unipotent characters in B and those in b.*

Proof. Consider the blocks B and b as above. We use Theorems 3.2 and 3.3. The correspondence between the quadratic unipotent characters in B and those in b is given by associating the character in B with parameters $(m_1, m_2, \rho_1, \rho_2)$ with the character in b with parameters $(h_1, h_2, \rho_1, \rho_2)$. This shows (ii).

For (i), let L be the Levi subgroup of the form $T_1 \times T_2 \times G_{n'}$ as in Theorem 3.1 (i). Then a defect group of B is isomorphic to an ℓ-Sylow subgroup of $(T_1 \rtimes (\mathbf{Z}_{2f} \wr S_{M_1})) \times (T_2 \rtimes (\mathbf{Z}_{2f} \wr S_{M_2}))$ (see [4], Theorem 22.9) for the unipotent block case, which extends to this case). By considering the Levi subgroup K of $Sp(2m, q)$ again as in Theorem 3.1, and noting that ℓ divides $q^f - 1$, we see that the defect group of b is isomorphic to the defect group of B. □

Corollary 4.1. *The map $B \to b$ as above gives a bijection from the set $\{\ell - \text{blocks of } U(n,q),\ \ell|(q^f-1)(f \text{ odd}),\ n \geq 0\}$ onto the set*

$$\{\ell - \text{blocks of } Sp(2m,q),\ \ell|(q^f-1)(f \text{ odd}),\ m \geq 0\}.$$

In order to consider the case of $GL(n,q)$ we prove the following lemma.

Lemma 4.1. *There is a natural bijection between*

$\{\ell\text{−blocks of } U(n,q),\ \ell|(q^f-1)(f \text{ odd})\}$ *and* $\{\ell\text{−blocks of } GL(n,q),\ \ell|(q^f+1)\ (f \text{ odd})\}$, *by Ennola Duality.*

Proof. The sets of quadratic unipotent characters of $GL(n,q)$ and $U(n,q)$ are in bijection via Ennola Duality, such that the characters in both groups parameterized by the same pair (μ_1, μ_2) correspond. (See e.g. ([1], 3.3) for the unipotent case, which extends to our case.) By [10] both the ℓ-blocks of $GL(n,q)$, $\ell|(q^f+1)$ (f odd) and the ℓ- blocks of $U(n,q)$, $\ell|(q^f -$

1) (f odd) are classified by $2f$-cores. Thus in both cases the blocks are parameterized by 6-tuples $(m_1, m_2, \sigma_1, \sigma_2, M_1, M_2)$. The map which makes the blocks of $GL(n, q)$ and $U(n, q)$ which are parameterized by the same 6-tuple correspond is then a bijection, which also induces a bijection of the quadratic unipotent characters in the blocks. □

Lemma 4.2. *There is a natural bijection between ℓ - blocks of H_n where $\ell|(q^f - 1)$, and ℓ - blocks where $\ell|(q^f + 1)$, by f-twisting. The quadratic unipotent characters in corresponding blocks also correspond by f-twisting. Here f is odd.*

Proof. By Lemma 3.1, if a symbol Λ is an f-core, then $\widehat{\Lambda}$ is an f-cocore. The ℓ - blocks of H_n where ℓ divides $q^f - 1$ (resp. $q^f + 1$) are classified by f-cores (resp. f-cocores). If b is an ℓ-block where ℓ divides $q^f - 1$ and b corresponds to a pair (π_1, π_2) of f-cores, let b^* be the ℓ-block where ℓ divides $q^f + 1$ which corresponds to the pair $(\widehat{\pi_1}, \widehat{\pi_2})$ of f-cocores.

The f-core (resp. cocore) of a symbol Λ is the f-twist of the f-cocore (resp. core) of the symbol $\widehat{\Lambda}$ (see [17], p.235). Thus there is a bijection between the quadratic unipotent characters in the blocks b and b^*, again by f-twisting. □

We then get the following theorem, analogous to Theorem 4.1, by Ennola duality and f-twisting.

Theorem 4.2. *Let ℓ divide $q^f + 1$, f odd. Let B be a quadratic unipotent block of $GL(n, q)$ and let B^* be the block of $U(n, q)$ corresponding to B by Lemma 4.1. Then consider the block b^* of $Sp(2m, q)$ corresponding to B^*. By Lemma 4.2 b^* corresponds, by f-twisting to an ℓ-block b of $Sp(2m, q)$ where ℓ divides $q^f + 1$, f odd. Then B and b correspond in the sense that (i) their defect groups are isomorphic, and (ii) there is a natural bijection between the quadratic unipotent characters in B and those in b.*

We now have the following corollary.

Corollary 4.2. *The above map then gives a bijection from the set* $\{\ell-$ blocks of $GL(n, q), \ell|(q^f + 1)\ (f \text{ odd}), n \geq 0\}$ *onto the set*

$$\{\ell - \text{blocks of } Sp(2m, q), \ell|(q^f + 1)\ (f \text{ odd}), m \geq 0\}.$$

We now consider the case where ℓ divides $q^f + 1$. where $e = 2f$, f even, so that $e \equiv 0 \pmod 4$.

Theorem 4.3. *Let ℓ divide q^f+1, f even. Let B be a quadratic unipotent block of G_n parameterized by $(m_1, m_2, \sigma_1, \sigma_2, M_1, M_2)$. Then there is a block b of H_m such that B and b correspond in the sense that (i) their defect groups are isomorphic, and (ii) there is a natural bijection between the quadratic unipotent characters in B and those in b.*

Proof. The quadratic unipotent characters in B are constituents of of $R_L^{G_n}(1\times\mathcal{E}\times\chi_{(\lambda_1,\lambda_2)})$, where L is a Levi subgroup of the form $T_1\times T_2\times G_{n'}$, T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order $q^{2f}-1$, and 1 (resp. $\mathcal{E}$) is the trivial character (resp. character of order 2) of T_1 (resp. T_2). Here the pair of partitions (λ_1, λ_2) corresponds to $(m_1, m_2, \sigma_1, \sigma_2)$ where (σ_1, σ_2) are f-cores, and we have a character $\chi_{(\pi_1,\pi_2)}$ of a group $H_{m'}$ corresponding to $(h_1, h_2, \sigma_1, \sigma_2)$. By the proof of Theorem 3.3, (π_1, π_2) are f-cores since (σ_1, σ_2) are f-cores. The character obtained from $\chi_{(\pi_1,\pi_2)}$ by f-twisting is of the form $\chi_{(\tau_1,\tau_2)}$, where the symbols τ_1, τ_2 are f-cocores. Let b be the ℓ-block of a group H_m corresponding to this character and M_1, M_2, i.e. the block b such that the quadratic unipotent characters in it are constituents of $R_K^{H_m}(1\times\mathcal{E}\times\chi_{(\tau_1,\tau_2)})$, where K is a Levi subgroup of the form $T_1\times T_2\times H_{m'}$, T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order q^f+1, and 1 (resp. $\mathcal{E}$) is the trivial character (resp. character of order 2) of T_1 (resp. T_2) (see Theorem 3.1(iii)). Then B and b correspond as required: For (i) the proof is as in Theorem 4.1. For (ii) we note that there is a bijection by f-twisting between the quadratic unipotent constituents of $R_K^{H_m}(1\times\mathcal{E}\times\chi_{(\tau_1,\tau_2)})$ and those of $R_K^{H_m}(1\times\mathcal{E}\times\chi_{(\pi_1,\pi_2)}$ (see [17], p.235). However, the quadratic unipotent constituents of the latter are in bijection with the quadratic unipotent characters in B, since (σ_1, σ_2) are the 2-quotients of (λ_1, λ_2). This proves the result. ∎

Summarizing, we have bijections between the following sets; we list them in the order in which they were constructed.

(i) $\{\ell-$blocks of $U(n,q)$, $\ell|(q^f-1)$ $(f$ odd$), n\geq 0\}\leftrightarrow\{\ell-$blocks of $Sp(2m,q)$, $\ell|(q^f-1)$ $(f$ odd$), m\geq 0\}$.

(ii) $\{\ell-$blocks of $GL(n,q)$, $\ell|(q^f+1)$ $(f$ odd$), n\geq 0\}\leftrightarrow\{\ell-$blocks of $Sp(2m,q)$, $\ell|(q^f+1)$ $(f$ odd$), m\geq 0\}$.

(iii) $\{\ell-$blocks of $U(n,q)$, $\ell|(q^f+1)$ $(f$ even$), n\geq 0\}\leftrightarrow\{\ell-$blocks of $Sp(2m,q)$, $\ell|(q^f+1)$ $(f$ even$), m\geq 0\}$.

(iv) $\{\ell-$blocks of $GL(n,q)$, $\ell|(q^f+1)$ $(f$ even$), n\geq 0\}\leftrightarrow\{\ell-$blocks of $Sp(2m,q)$, $\ell|(q^f+1)$ $(f$ even$), m\geq 0\}$.

5. Perfect Isometries

In this section we assume that all the blocks considered have abelian defect groups. We generalize the result on perfect isometries between unipotent blocks of [1] to quadratic unipotent blocks. We use the classification of blocks by e-cuspidal pairs and the description of characters in the blocks (see [4], 22.9; [19], 3.9).

We first describe the defect groups and their normalizers of the blocks under consideration (see [1], pp.46,50).

Case 1. $G = G_n$. Let B be a block of G as in Section 3, so that ℓ divides $q^{2f} - 1$. Let L be a Levi subgroup of the form $T_1 \times T_2 \times G_r$, where T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order $q^{2f} - 1$. The defect group of B is then a Sylow ℓ-subgroup of $T_1 \times T_2$.

Case 2. $G = H_n$. Let b be a block of G as in Section 3, so that ℓ divides $q^f - 1$ or $q^f + 1$. Let L be a Levi subgroup of the form $T_1 \times T_2 \times H_r$, where T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order $q^f - 1$ or $q^f + 1$. The defect group of b is then a Sylow ℓ-subgroup of $T_1 \times T_2$.

We note that the defect groups of two blocks B and b which correspond as in Section 4 are isomorphic.

In each case, we have $W_G(L) = N_G(L)/L \cong \mathbf{Z}_{2f} \wr S_{M_1+M_2}$, where S_N is the symmetric group of degree N. Now suppose λ is a quadratic unipotent $2f$-cuspidal character (resp. f-cuspidal character) of G_r (resp. H_r). Then we have in each case $W_G(L, \lambda) = N_G(L, \lambda)/L = W_1 \times W_2$ where $W_1 \cong \mathbf{Z}_{2f} \wr S_{M_1}$ and $W_2 \cong \mathbf{Z}_{2f} \wr S_{M_2}$.

The results of Broué, Malle and Michel ([1], 3.2, 5.15) can be modified as follows.

Theorem 5.1. *Let $G = G_n$ or H_n and L a Levi subgroup of G as in Case 1 or Case 2 above. Let λ be a quadratic unipotent character of L of the form $1 \times \mathcal{E} \times \chi$, where 1 is a trivial character (resp. character of order 2) of T_1 (resp. T_2), and χ is in a block of defect 0 of G_r or H_r, so that (L, λ) is an e-cuspidal pair in Case 1 and an f-cuspidal pair in Case 2.*

Let M be an $2f$-split Levi subgroup containing L in Case 1 or an f-split or $2f$-split Levi subgroup containing L in Case 2. We then have an isometry $I^M_{(L,\lambda)}$ between the $\mathbf{Z}$-spans of the set $\mathrm{Irr}(W_M(L, \lambda))$ and the set of constituents of $R^M_L(\lambda)$ such that $R^G_M \, . \, I^M_{(L,\lambda)} = I^G_{(L,\lambda)} \, . \, \mathrm{Ind}^{W_G(L,\lambda)}_{W_M(L,\lambda)}$.

Proof. If $G = G_n$ (resp. H_n) the quadratic unipotent characters are of the form $\chi_{(\mu_1,\mu_2)}$ (resp. $\chi_{(\Lambda_1,\Lambda_2)}$) where μ_1, μ_2 are partitions and Λ_1, Λ_2 are symbols. In this case the characters are in a fixed Lusztig series and thus in bijection with the unipotent characters of the centralizer of a semisimple element. Thus we have fixed integers n_1, n_2 such that $n_1 + n_2 = n$, and μ_1, μ_2 are partitions of n_1, n_2 respectively and Λ_1, Λ_2 are symbols of rank n_1, n_2 respectively.

In the case of the unipotent characters of G_n and H_n the group M has been described in ([1]. p.46, p.49-52). From our choice of f the group M in our case can be assumed to have the following form. In the case of G_n, $M = GL(b, q^{2f}) \times G_k$ for some b, k, and in the case of H_n, $M = GL(b, q^f) \times H_k$ or $M = U(b, q^f) \times H_k$ for some b, k. We have $b \leq M_1 + M_2$.

Suppose L is embedded in M as follows. Let $T_1 = T_{1,1} \times T_{1,2}$, $T_2 = T_{2,1} \times T_{2,2}$.

Case 1. Let $T_{1,1} \times T_{2,1} \subseteq GL(b, q^f)$, $T_{1,2} \times T_{2,2} \times G_r \subseteq G_k$, where $T_{1,1}$ (resp. $T_{2,1}$) is isomorphic to b_1 (resp. b_2) copies of tori of orders $q^{2f} - 1$.

Case 2. Let $T_{1,1} \times T_{2,1} \subseteq GL(b, q^f)$ or $U(b, q^f$, $T_{1,2} \times T_{2,2} \times H_r \subseteq H_k$, where $T_{1,1}$ (resp. $T_{2,1}$) is isomorphic to b_1 (resp. b_2) copies of tori of orders $q^f - 1$ or $q^f + 1$.

Recall that $W_G(L, \lambda) = N_G(L, \lambda)/L = W_1 \times W_2 \cong \mathbf{Z}_{2f} \wr S_{M_1} \times \mathbf{Z}_{2f} \wr S_{M_2}$.

Since the character λ takes the value 1 on T_1 and $\mathcal{E}$ on T_2, we see that for both G_n and H_n we get $W_M(L, \lambda) = W_1{}' \times W_2{}'$ where $W_1{}' \cong S_{b_1} \times (\mathbf{Z}_{2f}) \wr S_{M_1 - b_1}$ and $W_2{}' \cong S_{b_2} \times (\mathbf{Z}_{2f}) \wr S_{M_2 - b_2}$.

Now we consider Lusztig induction $R_L^M(\lambda)$ where λ is of the form $\chi_{(\lambda_1, \lambda_2)}$, where (λ_1, λ_2) is a pair of partitions or symbols. Using results of Waldspurger [21] it was shown in ([20], 4.2) for the case of H_n that Lusztig induction commutes with Jordan decomposition. More precisely, we have: The constituents of $R_L^M(\lambda)$ are of the form $\chi_{(\mu_1, \mu_2)}$, where the μ_i are obtained from the λ_i by adding a succession of hooks or cohooks.

We consider the case of H_n. Then $C_{G^*}(s) = K_1 \times K_2$ where K_1 (resp. K_2) is isomorphic to $SO(2m_1 + 1)$ (resp. $O^{\pm 1}(2m_2)$) for some m_1, m_2 with $m_1 + m_2 = n$. We have subgroups M^*, L^* which are intersections of subgroups dual to M, L with $C_{G^*}(s)$. Then we have $M^* = M_1 \times M_2$, $L^* = L_1 \times L_2$, where $M_1, L_1 \subseteq K_1$ and $M_2, L_2 \subseteq K_2$, and characters λ_i of L_i, $i = 1, 2$. By applying ([1], 3.2) to the groups K_i we have isometries between the $\mathbf{Z}$-spans of the set $\mathrm{Irr}(W_{M_i}(L_i, \lambda_i))$ and the set of constituents of $R_{L_i}^{M_i}(\lambda_i)$ such that $R_{M_i}^{K_i} \,.\, I_{L_i, (\lambda_i)}^{M_i} = I_{(L_i, \lambda_i)}^{K_i} \,.\, \mathrm{Ind}_{W_{M_i}(L_i, \lambda_i)}^{W_{K_i}(L_i, \lambda_i)}$, $i - 1, 2$.

We now define $I_{(L,\lambda)}^M$ as follows. Let $(\psi_1, \psi_2) \in \mathrm{Irr}(W_M(L, \lambda) = W_1{}' \times W_2{}')$. We identify $W_i{}'$ with $W_{M_i}(L_i, \lambda_i)$. Suppose $I_{(L_i, \lambda_i)}^{K_i}(\psi_i) = \chi_{\mu_i}$, a constituent of $R_{L_i}^{M_i}(\lambda_i)$. Then define $I_{(L,\lambda)}^M((\psi_1, \psi_2)) = \chi_{(\mu_1, \mu_2)}$. We then have an isometry $I_{(L,\lambda)}^M$ between the $\mathbf{Z}$-spans of the set $\mathrm{Irr}(W_M(L, \lambda))$ and the set of constituents of $R_L^M(\lambda)$ such that $R_M^G \cdot I_{(L,\lambda)}^M = I_{(L,\lambda)}^G \cdot \mathrm{Ind}_{W_M(L,\lambda)}^{W_G(L,\lambda)}$. This proves the theorem. □

The proof of Theorem 5.1 is a formal extension of ([1], 3.2). We now give an explicit description of the maps $I_{(L,\lambda)}^G$ in our case, as in ([1], p.50).

In the case of $G = H_n$ and $G = U(n,q)$, the parametrization of quadratic unipotent characters by 4-tuples $(h_1, h_2, \rho_1, \rho_2)$ and by 4-tuples $(m_1, m_2, \rho_1, \rho_2)$ respectively arises from their construction by Lusztig [13] by Harish-Chandra induction. Consider the characters occurring in $R_L^G(\lambda)$ for appropriate (G, L, λ). The description given in ([1], p.50) shows that, given such a character, each ρ_i corresponds to a $2f$-tuple of partitions whose sizes add up to M_i, $i = 1, 2$. (Here the M_i are weights, denoted by a in op.cit. where the characters are unipotent.) Since the irreducible characters of $W_G(L,\lambda)$ are parametrized by $2f$-tuple of partitions, this defines the map $I_{(L,\lambda)}^G$ in this case. The case of $G = GL(n,q)$ follows from that of $G = U(n,q)$.

Then we have a bijection with signs between the set of quadratic unipotent characters occurring in $R_L^G(\lambda)$ and the set $\mathrm{Irr}(W_G(L,\lambda))$. We then see that the character of G_n parametrized by $(m_1, m_2, \rho_1, \rho_2)$ and the character of H_n parametrized by $(h_1, h_2, \rho_1, \rho_2)$ correspond to the same character in $\mathrm{Irr}(W_G(L,\lambda))$ in the above bijection, where we choose G, L, λ appropriately in each case.

Now consider the case of $G = H_n$ where λ corresponds to a pair of symbols which are f-cocores. By f-twisting we have a bijection between the quadratic unipotent characters occurring in $R_L^G(\lambda)$ and those occurring in $R_{L'}^G(\lambda')$ where (L', λ') are as in the previous case, i.e. with λ' corresponding to a pair of symbols which are f-cores. We can compose this bijection with the bijection of the previous case.

We thus have:

Theorem 5.2. *Let B and b be blocks with abelian defect groups of a pair G_n and H_m which correspond as in Section 4, Theorems 4.1, 4.2, 4.3. Then the correspondence between the sets of quadratic unipotent characters in B and b factors through the isometry of these sets with the sets $\mathrm{Irr}(W_G(L,\lambda))$ with appropriate G, L, λ for G_n and H_m.*

Next we consider perfect isometries, and an analog of ([1], 5.15). For this we need to consider characters $\theta \in \mathrm{Irr}(Z(L)_\ell)$ for L a Levi subgroup of $G = G_n$ or $G = H_n$ as in Theorem 3.1 (in the case of H_n this subgroup was denoted by K). In ([1], 5.15) a subgroup $G(\theta)$ of G has been introduced. Here we give an alternative definition of this group, analogous to a definition in ([5], p.163). Consider a subgroup L^* of G^* in duality with L, then an ℓ-element $t \in (Z(L^*)_\ell)$. Then $C_{G^*}(t)^0$ is a Levi subgroup of G^* and there is a subgroup $G(t)$ of G in duality with $C_{G^*}(t)^0$. Since ℓ is odd $G(t)$ is isomorphic to $G(\theta)$, where θ corresponds to a linear character $\hat{t}$ of $G(t)$, defined when we have chosen a fixed embedding of $\overline{\mathbf{F}}_q^*$ into $\overline{\mathbf{Q}}_l$. We will use the subgroup $G(t)$ instead of $G(\theta)$ in the following. The groups $G(t)$ can be explicitly described as being isomorphic to $\prod_i GL(m_i, q^{2f}) \times G_r$ or $\prod_i U(m_i, q^{2f}) \times G_r$ in the case of G_n, and to $\prod_i GL(m_i, q^f) \times H_r$ or $\prod_i U(m_i, q^f) \times H_r$ in the case of H_n.

We consider a quadratic unipotent block b of $G = G_n$ or H_n. We have seen that the quadratic unipotent characters in b are constituents of $R_L^G(1 \times \mathcal{E} \times \chi_{(\pi_1,\pi_2)})$, where L is a suitable Levi subgroup and (π_1, π_2) are $2f$-core partitions or f-core or f-cocore symbols. We now consider the other characters in b. As in ([5], p.163) we get that a character in b is of the form $R_{G(t)}^G(\hat{t}\chi)$, up to sign, where χ is a quadratic unipotent character of G. We note that in this case $R_{G(t)}^G$ is an isometry, which is a special case of ([5], p.163). We also note that an irreducible character of $Z(L)_\ell \ltimes W_G(L,\lambda)$ can be written as $\hat{t}\tau$ for some $t \in (Z(L^*)_\ell)$ and an irreducible character τ of $W_G(L,\lambda)$ as in ([1],p.71). We now state the analog of ([1], 5.15) in our case.

Theorem 5.3. *Let $G = G_n$ or $G = H_n$.The map*

$$I_{(L,\lambda)}^G : \mathbf{Z}\mathrm{Irr}(Z(L)_\ell \ltimes W_G(L,\lambda)) \to \mathbf{Z}\ \mathrm{Irr}(G,b)$$

such that

$$\mathrm{Ind}_{W_{G(t)}(L,\lambda)}^{W_G(L,\lambda)}(\hat{t}\tau) \to R_{G(t)}^G(\hat{t}I_{(L,\lambda)}^{G(t)}(\tau)$$

is an ℓ-perfect isometry between $(Z(L)_\ell \ltimes W_G((L,\lambda), b(1.(1\times\mathcal{E})))$ and (G,b).

Here we interpret the character $1.(1 \times \mathcal{E})$ as follows. We have $W_G(L,\lambda) = W_1 \times W_2$ as in Theorem 5.1. We take the trivial character 1 on $Z(L)_\ell$, the character 1 on W_1 and the character $\mathcal{E}$ on W_2. Then $b(1.(1 \times \mathcal{E})))$ is the block containing $1.(1 \times \mathcal{E})$ of $(Z(L)_\ell \ltimes W_G((L,\lambda)$.

Proof. We use the definition of ℓ-perfect isometry given in ([1], 5.11). We note the following points in the proof of ([1], 5.15) at which unipotent characters have to be replaced by quadratic unipotent characters.

- The f-Harish-Chandra theory was proved for quadratic unipotent characters in classical groups in ([19]), which gives us the analog of ([1], 5.19, 5.18).
- We have verified the extension to our case of ([1]), 3.2) in Theorem 5.1. This is used in ([1], 5.17).
- An e-cuspidal or f-cuspidal quadratic unipotent character is of defect 0 for $G = G_n$ or $G = H_n$. This follows by Jordan decomposition and by degree considerations. This generalizes ([1], 5.21).

Then the proof is formally completely analogous to that of ([1], 5.15). Part (ii) of the result shows that there is an ℓ-perfect isometry between $(Z(L)_\ell \ltimes W_G(L,\lambda), 1.(1\times\mathcal{E}))$ and (G,b). $\square$

We now consider the groups G_n and H_n.

Theorem 5.4. *We have ℓ-perfect isometries in the sense of ([1], 5.11) between the* **Z***-spans of the characters in corresponding blocks of the following groups;*

(i) An $\ell-$block of $GL(n,q)$, $\ell|(q^f+1)$ (f odd) *and an* $\ell-$block of $Sp(2m,q)$, $\ell|(q^f+1)$ (f odd),

(ii) An $\ell-$block of $U(n,q)$, $\ell|(q^f-1)$ (f odd) *and an* $\ell-$block of $Sp(2m,q)$, $\ell|(q^f-1)$ (f odd) *.*

(iii) An $\ell-$block of $GL(n,q)$, $\ell|(q^f+1)$ (f even) *, and an* $\ell-$block of $Sp(2m,q)$, $\ell|(q^f+1)$ (f even).

(iv) An $\ell-$block of $U(n,q)$, $\ell|(q^f+1)$ (f even) *and an* $\ell-$block of $Sp(2m,q)$, $\ell|(q^f+1)$ (f even) *.*

In cases (i) and (ii), the block parameterized by $(m_1,m_2,\sigma_1,\sigma_2,M_1,M_2)$, where $m_1(m_1+1)/2+m_2(m_2+1)/2+2N_1+2N_2=n$, corresponds to the block parameterized by $(h_1,h_2,\sigma_1,\sigma_2,M_1,M_2)$, where $h_1(h_1+1)+h_2^2+N_1+N_2=m$. In cases (iii) and (iv) the blocks correspond as in Theorem 4.3.

Proof. The theorem follows from Theorem 5.3, since in each case there is a perfect isometry between the blocks in question and a block of a “local” group of the form $Z(L)_\ell \ltimes W_G(L,\lambda)$. $\square$

Theorem 5.5. *Suppose a block B of G_n and a block b of H_n correspond as in Theorem 5.4. The quadratic unipotent characters in B and b correspond under the isometry as follows: In cases (i) and (ii) above, the character of G_n parametrized by (m_1,m_2,ρ_1,ρ_2) corresponds to the character of H_n parametrized by (h_1,h_2,ρ_1,ρ_2). In cases (iii) and (iv) the characters correspond as in Theorem 4.3.*

Proof. The theorem follows from the fact that in the map $I^G_{(L,\lambda)}$ in Theorem 5.3 we can take $t=1$. Using Theorem 5.2 we get the correspondence between characters as in Theorem 2.1. $\square$

Remark. The case of G_n is easier than that of H_n, as is seen below.

Let $G=G_n$, B a quadratic unipotent block of G_n. The quadratic unipotent characters in B are of the form $\chi_{(\mu_1,\mu_2)}$in the Lusztig series $\mathcal{E}(G,(s))$, where (μ_1,μ_2) are partitions of a fixed pair k_1,k_2 respectively. By a result of Bonnafé and Rouquier the block B is Morita equivalent to a block $B(s)$ of $C_G(s)$. Now since s is central in $C_G(s)$ the block $B(s)$ can be regarded as the product of two unipotent blocks of $C_G(s)$, and thus ([1],5.15) can be applied to it. We get a perfect isometry between the block and a quadratic unipotent block of the "local subgroup" $Z(L)_\ell \rtimes W_G(L,\lambda)$.

We now consider signs appearing in the perfect isometries of Theorem 5.4 and Theorem 5.5. Consider a quadratic unipotent character χ of G_n parametrized by a pair (λ_1, λ_2) of partitions which corresponds to the quadratic unipotent character ψ of H_m parametrized by a pair (Λ_1, Λ_2) of symbols under the perfect isometry. Enguehard ([9], p.34) has used the combinatorics of partitions and symbols to define a sign ν_e on partitions and symbols and uses them to calculate the signs which appear in ([9], Theorem B), which is the same theorem as ([1], 3.2). Thus the sign appearing in the correspondence between χ and ψ as above is $\nu_e(\lambda_1)\nu_e(\lambda_2)\nu_e(\Lambda_1)\nu_e(\Lambda_2)$.

6. Endoscopic groups

Let G be a finite reductive group, ℓ a prime as before, and (s) an ℓ-prime semisimple class in G^*. Let B be an ℓ-block of G parameterized by (s). M.Enguehard has proved the following [8]. There is a (possibly disconnected) group $G(s)$ which need not be a subgroup of G, and a block $B(s)$ of $G(s)$ such that B and b correspond, in the following sense:

- There is a bijection between characters in B and $B(s)$
- The defect groups of B and $B(s)$ are isomorphic
- The Brauer categories of B and $B(s)$ are equivalent

The group $G(s)$ is dual to the centralizer of s in G^*. We call $G(s)$ an endoscopic group of G, in analogy with a terminology used in p-adic groups. We describe the endoscopic groups in our case ([8], 3.5.4).

Case 1. $G = G_n$, B corresponds to the Levi subgroup L of the form $T_1 \times T_2 \times G_{n'}$, T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order $q^{2f} - 1$, and we take a character of L to be 1 (resp. $\mathcal{E}$) on T_1 (resp. T_2) and the character $\chi_{(\lambda_1,\lambda_2)}$ of defect 0 of $G_{n'}$. The pair (λ_1, λ_2) corresponds to a pair (m_1, m_2) as before. Then $s \in G_n = G_n^*$ has n_1 (resp. n_2) eigenvalues 1 (resp. -1) where $n_1 = 2fM_1 + |\lambda_1|$, $n_2 = 2fM_2 + |\lambda_2|$.

Then $G(s) = G_n(s) \cong G_{n_1} \times G_{n_2}$.

Case 2. $G = H_m$, B corresponds to the Levi subgroup L of the form $T_1 \times T_2 \times H_{m'}$, T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order $q^f - 1$ or $q^f + 1$, and we take a character of L to be 1 (resp. $\mathcal{E}$) on T_1 (resp. T_2) and the character $\chi_{(\pi_1,\pi_2)}$ of defect 0 of $H_{m'}$. The pair (π_1, π_2) corresponds to a pair (h_1, h_2) as before. Then $s \in H_m^*$ has k_1 (resp. k_2) eigenvalues 1 (resp. -1) where $k_1 = fM_1 + \operatorname{rank}\pi_1$, $k_2 = fM_2 + \operatorname{rank}\pi_2$. We note that $H_m^* \cong SO(2m+1)$.

Then $H(s) = H_m(s) \cong Sp(2k_1, q) \times O(2k_2, q)$. Here we get $O^+(2k_2, q)$ if h_2 is even and $O^-(2k_2, q)$ if h_2 is odd (see [21], 4.3).

Under the Jordan decomposition of characters, the quadratic unipotent characters of G_n and H_m correspond to characters of $G_n(s)$ and $H_m(s)$ respectively which are tensor products of unipotent characters with a fixed linear character $\hat{s}$. There is a bijection between the set of quadratic unipotent blocks of G_n (resp. H_m) and the set of blocks of $G_n(s)$ (resp. $H_m(s)$) which contain the characters as above, and then a bijection between the set of quadratic unipotent blocks of G_n (resp. H_m) and the set of unipotent blocks of $G_n(s)$ (resp. $H_m(s)$) . The proof of the theorem below follows from these bijections.

Theorem 6.1. *We have block correspondences between unipotent blocks of endoscopic groups as follows. As in Theorems 4.1,4.2,4.3 we have (i) the defect groups of corresponding blocks B and b are isomorphic, and (ii) there is a natural bijection between the unipotent characters in B and those in b.*

$\{\ell-$ blocks of $GL(n_1, q) \times GL(n_2, q),\ \ell|(q^f + 1)\ (f$ odd$),\ n \geq 0\} \leftrightarrow \{\ell-$ blocks of $Sp(2k_1, q) \times O(2k_2, q),\ \ell|(q^f + 1)\ (f$ odd$),\ m \geq 0\}$.

$\{\ell-$ blocks of $U(n_1, q) \times U(n_2, q),\ \ell|(q^f - 1)\ (f$ odd$),\ n \geq 0\} \leftrightarrow \{\ell-$ blocks of $Sp(2k_1, q) \times O(2k_2, q), \ell|(q^f - 1)\ (f$ odd$),\ m \geq 0\}$

$\{\ell-$ blocks of $U(n_1, q) \times U(n_2, q),\ \ell|(q^f + 1)\ (f$ even$), n \geq 0\} \leftrightarrow \{\ell-$ blocks of $Sp(2k_1, q) \times O(2k_2, q),\ \ell|(q^f + 1)\ (f$ even$), m \geq 0\}$

$\{\ell-$ blocks of $GL(n_1, q) \times GL(n_2, q),\ \ell|(q^f + 1)\ (f$ even$),\ n \geq 0\} \leftrightarrow \{\ell-$ blocks of $Sp(2k_1, q) \times O(2k_2, q),\ \ell|(q^f + 1)\ (f$ even$),\ m \geq 0\}$

Here $n = n_1 + n_2$ and $m = k_1 + k_2$ correspond as before, and n_1, n_2, k_1, k_2 are as defined.

We now consider perfect isometries between the corresponding blocks above, which follow easily from the case of [1].

Let $B(s)$ be an ℓ-block of $G_n(s) = G_1 \times G_2$, where $G_1 = G_{n_1}$ and $G_2 = G_{n_2}$. Then $B(s)$ factorizes as $B_1(s) \times B_2(s)$ where $B_1(s)$ and $B_2(s)$ are blocks of G_1 and G_2 respectively. There are Levi subgroups $L_1(s)$ and $L_2(s)$ of G_1 and G_2 respectively such that $L_1(s) = T_1 \times G_{n'_1}$ and $L_2(s) = T_2 \times G_{n'_2}$. Here T_1 (resp. T_2) is a product of M_1 (resp. M_2) tori of order $q^{2f} - 1$. Consider the "local group" $((T_1)_\ell \times (T_2)_\ell) \ltimes (\mathbf{Z}_{2f} \wr S_{M_1} \times \mathbf{Z}_{2f} \wr S_{M_2})$. A character θ of $(T_1)_\ell \times (T_2)_\ell$ factorizes as $\theta_1 \times \theta_2$, where $\theta_i \in \mathrm{Irr}((T_i)_\ell)$, $i = 1, 2$. Then the pair θ_1, θ_2 determines a pair (t_1, t_2) of ℓ-elements in $G_1 \times G_2$, and then a subgroup $G(t_1) \times G(t_2)$ of $G_1 \times G_2$ which plays a role analogous to that of $G(t)$ in the case of G_n. Since $B(s)$ is a product of blocks of G_1 and G_2 containing characters which are products of a fixed linear character and *unipotent* characters, by an application of [1] we get a perfect isometry of $(G_n(s), B(s))$ with the principal block of the "local group" $((T_1)_\ell \times (T_2)_\ell) \ltimes (\mathbf{Z}_{2f} \wr S_{M_1} \times \mathbf{Z}_{2f} \wr S_{M_2})$.

In the case of $(H_m(s), b(s))$, similarly we get a perfect isometry with the principal block of the same "local group" $(T_1)_\ell \times (T_2)_\ell \ltimes (\mathbf{Z}_{2f} \wr S_{M_1} \times \mathbf{Z}_{2f} \wr S_{M_2})$. We note that here the elements t_1, t_2 are to be taken in the dual group ${H_m}^*$. We also note that in the case where we have a group of the form $O(2k, q)$ we use results of Malle [16] extending the results of [1] to disconnected groups. Finally we get a perfect isometry between $B(s)$ and $b(s)$.

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