Modular representations of general linear groups: An Overview

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To Robert Steinberg on his 90th birthday
The outline of my talk

- Ordinary Representations of $GL_n$
- Combinatorics of tableaux
- Modular Representations of $GL_n$
- Blocks
- Decomposition Numbers
- New Methods: Lie Theory
Ordinary representation Theory over $K$ of characteristic 0 
Modular Representation Theory over $k$ of characteristic $\ell$ not dividing $q$

- a partition of the ordinary characters, or $KG$-modules, into blocks
- a partition of the Brauer characters, or $kG$-modules, into blocks
- a partition of the decomposition matrix into blocks
Some main problems of modular representation theory:

- Describe the blocks as sets of characters, or as algebras
- Describe the irreducible modular representations, e.g. their degrees
- Find the decomposition matrix $D$, the transition matrix between ordinary and Brauer characters.
- Global to local: Describe information on the block $B$ by "local information", i.e. from blocks of subgroups of the form $N_G(P)$, $P$ a $p$-group
$G = GL(n, q)$ has subgroups:

- Tori, abelian subgroups (e.g. diagonal matrices)
- Levi subgroups, products of subgroups of the form $GL(m, q^d)$
- Borel subgroups, isomorphic to “upper triangular matrices”
- Parabolic subgroups of the form $P = LV$, $L$ a product of subgroups of the form $GL(m, q)$, $V \triangleleft P$
Parabolic subgroup $P$ is of the form

$$
\begin{pmatrix}
\clubsuit & * & * & \ldots & * \\
0 & \clubsuit & * & \ldots & * \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \clubsuit
\end{pmatrix}
$$
Then $L$ is of the form
\[
\begin{pmatrix}
\clubsuit & 0 & 0 & \cdots & 0 \\
0 & \clubsuit & 0 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \clubsuit
\end{pmatrix}
\]
And $V$ is of the form

$$\begin{pmatrix}
I & * & * & \ldots & * \\
0 & I & * & \ldots & * \\
. & . & . & . & . \\
0 & 0 & 0 & 0 & I
\end{pmatrix}$$
Examples of tori:

- Subgroup of diagonal matrices
- Cyclic torus generated by

$$
\begin{pmatrix}
\alpha & 0 & \cdots & 0 \\
0 & \alpha^q & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \alpha^{q^{n-1}}
\end{pmatrix}
$$

($\alpha$ is a primitive $q^n - 1$-th root of unity)
- products of cyclic tori as above
Harish-Chandra theory (applicable to finite reductive groups)

$P$ a parabolic subgroup of $G$, $L$ a Levi subgroup of $P$, so that $L \leq P \leq G$.

Harish-Chandra induction is the following map:

$$R_L^G : K_0(KL) \to K_0(KG).$$

If $\psi \in \text{Irr}(L)$ then $R_L^G(\psi) = \text{Ind}_P^G(\tilde{\psi})$ where $\tilde{\psi}$ is the character of $P$ obtained by inflating $\psi$ to $P$. 
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$\chi \in \text{Irr}(G)$ is cuspidal if $\langle \chi, R^G_L(\psi) \rangle = 0$ for any $L \leq P < G$ where $P$ is a proper parabolic subgroup of $G$. The pair $(L, \theta)$ a cuspidal pair if $\theta \in \text{Irr}(L)$ is cuspidal.

**THEOREM.** (i) Let $(L, \theta), (L', \theta')$ be cuspidal pairs. Then $\langle R^G_L(\theta), R^G_{L'}(\theta') \rangle = 0$ unless the pairs $(L, \theta), (L', \theta')$ are $G$-conjugate. (ii) If $\chi$ is a character of $G$, then $\langle \chi, R^G_L(\theta) \rangle \neq 0$ for a cuspidal pair $(L, \theta)$ which is unique up to $G$-conjugacy.

$\text{Irr}(G)$ partitioned into Harish-Chandra families: A family is the set of constituents of $R^G_L(\theta)$ where $(L, \theta)$ is cuspidal.
Example: If $L$ is the subgroup of diagonal matrices contained in the (Borel) subgroup $B$ of upper triangular matrices, do Harish-Chandra induction, i.e. lift a character of $L$ to $B$ and do ordinary induction.

But if $L$ is a Levi subgroup not in a parabolic subgroup, e.g. a torus of order $q^n - 1$, we must do Deligne-Lusztig induction to obtain generalized characters from characters of $L$. 
Deligne-Lusztig Theory (applicable to finite reductive groups)

Suppose $L$ is a Levi subgroup, not necessarily in a parabolic subgroup $P$ of $G$.

The Deligne-Lusztig linear operator:

$$R^G_L : K_0(\overline{Q}L) \rightarrow K_0(\overline{Q}G).$$

$R^G_L$ takes (ordinary) characters of $L$ to $\mathbb{Z}$-linear combinations of characters of $G$.

A unipotent character is a constituent of $R^G_T(1)$, $T$ a maximal torus.
Unipotent classes indexed by partitions of $n$ (Jordan form).

*unipotent characters* of $G$ are constituents of $\text{Ind}_B^G$ (not true in general)

Also indexed by partitions of $n$, denoted by $\chi_\lambda$, $\lambda$ a partition of $n$. 
Example: Unipotent characters of $GL(3, q)$, constructed by Harish-Chandra induction, values at unipotent classes

[R. Steinberg, Canadian J. Math. 3 (1951)]

<table>
<thead>
<tr>
<th>$\chi[4]$</th>
<th>1</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi[31]$</td>
<td>$q^2 + q$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\chi[1^3]$</td>
<td>$q^3$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Recall the blocks of $S_n$, described by $p$-cores.

**Theorem (Brauer-Nakayama)** $\chi_\lambda, \chi_\mu$ are in the same $p$-block if and only if $\lambda, \mu$ have the same $p$-core.
\( \ell \) a prime not dividing \( q \), \( e \) the order of \( q \) mod \( \ell \)

Theorem (Fong-Srinivasan) \( \chi_\lambda, \chi_\mu \) are in the same \( \ell \)-block if and only if \( \lambda, \mu \) have the same \( e \)-core.

Example: \( n = 5 \), \( \ell \) divides \( q^2 + q + 1 \), \( e = 3 \). Then \( \chi_\lambda \) for \( 5, 2^21, 21^3 \) are in a block. Same for \( S_5, p = 3 \).

Example: \( n = 4 \), \( \ell \) divides \( q^2 + 1 \), \( e = 4 \). Then \( \chi_\lambda \) for \( 4, 31, 21^2, 1^4 \) are in a block.

\( \quad \) has no 4-hooks.
The ordinary characters in a block can be described via Deligne-Lusztig induction. Brauer meets Lusztig!
Work done on blocks and decomposition matrices for classical groups: Dipper-James, Geck, Gruber, Hiss, Kessar, Malle ...

Describe the unipotent part of the $\ell$-modular decomposition matrix of $G$.

Can write $\chi_\lambda = \sum d_{\lambda \mu} \phi_\mu$ where $\phi_\mu$ are Brauer characters.

Describe $d_{\lambda \mu}$.
$M$ the module induced to $G$ from the trivial character of $B$, $B =$ Borel=upper triangular matrices

$\text{End}_G(M)$ is isomorphic to The Hecke algebra $H_n$ of type $A$. Has generators $\{T_1, T_2, \ldots, T_{n-1}\}$ and some relations, e.g.

$T^2_i = (q - 1)T_i + q.1.$

When $H_n$ is not semisimple ($q$ a root of unity) we can talk of its modular representations, blocks, decomposition numbers, etc.

Over a field $F$, define Specht modules $S_\lambda$, irreducible modules $L_\lambda$, $\lambda$ partition of $n$. Then we want the multiplicity $(S_\lambda : L_\mu)$ (as for $S_n$).
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A new object: The $q$-Schur algebra $S_q(n)$ can be defined over any field, as the endomorphism algebra of a $H_n$-module.

$\text{char} k = \ell$. Then $S_q(n) = \text{End}_{H_n} \oplus M_\lambda$, $M_\lambda$ are certain permutation $H_n$-modules.

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$S_q(n)$ and $H_n$ are in $q$-Schur-Weyl duality!
$S_q(n)$ has Weyl modules, simple modules analogous to Specht modules, simple modules for $H_n$ or $S_n$.

Here $S_q(n)$ is over $k$ and we can talk of its decomposition numbers, from its Weyl modules and simple modules over $k$, as $(W_\lambda : L_\lambda)$.

Theorem (Dipper-James) The decomposition matrix of $S_q(n)$ over a field of characteristic $\ell$ is the same as the unipotent part of the decomposition matrix of $G$. 
An example of a decomposition matrix $D$ for $n = 4$, $e = 4$:

\[
\begin{pmatrix}
4 & 1 & 0 & 0 & 0 \\
31 & 1 & 1 & 0 & 0 \\
211 & 0 & 1 & 1 & 0 \\
1111 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\]
New modular representation theory connects decomposition numbers for symmetric groups, Hecke algebras, $q$-Schur algebras, with Lie theory.

The quantized Kac-Moody algebra $U_v(\widehat{sl}_e)$ over $\mathbb{Q}(v)$ is generated by $e_i, f_i, k_i, k_i^{-1}, \ldots, (0 \leq i \leq e - 1)$ with some relations.
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Fock space: $\mathbb{Q}(\nu)$-vector space with basis $s_\lambda$ where $\lambda$ runs over the partitions of $n \geq 0$. Can think of the $s_\lambda$ as indexing
(1) the Weyl modules of $S_q(n)$ over a field of characteristic 0 with $q$ a primitive $e$-th root of unity, $n \geq 0$
(2) unipotent characters of $GL(n, q)$, $n \geq 0$
Then $\mathcal{U}_v(\widehat{sl}_e)$ acts on the Fock space! $e_i, f_i$ are functors on the Fock space, called $i$-induction, $i$-restriction (as in the case of $S_n$). Blocks appear as weight spaces for the subalgebra generated by the $k_i$.

Work of Ariki, Grojnowski, Vazirani, Lascoux-Leclerc-Thibon, Varagnolo-Vasserot, ...
Decomposition matrix $D$ for the $q$-Schur algebra $S_q(n)$ over $K$ with $q$ an $e$-th root of unity, entries $d_{\lambda\mu}$, can be determined in principle. This does not give $D$ for $G$, as the Dipper–James Theorem is for characteristic $\ell$. 
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Summary

- Known: Decomposition numbers for $H_n$ (also cyclotomic Hecke algebras) over characteristic 0
- Known: Decomposition numbers for $GL_n(q)$, $\ell$ large
- Not known: Decomposition numbers for $S_n$, $GL_n(q)$, all $\ell$
References:


A. Mathas, Iwahori Hecke Algebras and Schur algebras of the symmetric group, University Lecture Series 15, AMS (1999).

B. Srinivasan, Modular Representations, old and new, in Springer PROM (2011)