The BMM Global-Local bijection for GL(n,q)

Bhama Srinivasan

University of Illinois at Chicago

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Let $G_n = GL(n, q)$, ℓ a prime not dividing q, e the order of q mod ℓ.

Unipotent characters of G_n are constituents of $\mathrm{Ind}_B^{G_n}(1)$ (B a Borel) and are indexed by partitions of n.

Denoted by χ_λ, λ a partition of n.

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Alternatively: Unipotent blocks classified by pairs (λ, k) (e-core, weight)

If $B \leftrightarrow (\lambda, k)$, then $\chi_\mu \in B$ iff χ_μ is a constituent of $< R_L^{G_n}(\chi_\lambda)$, where (L, χ_λ) is an e-cuspidal pair

L (e-split Levi) is isomorphic to a product of k copies of tori of order $q^e - 1$ and G_m, G_m has e-cuspidal χ_λ.

$N(L)/L$ isomorphic to $W(L, \lambda) = \mathbf{Z}_e \wr S_k = G(e, 1, k)$

Broué, Malle, Michel: Global to Local Bijection for G_n:

Isometry I_L^G maps ϕ_{μ^*}, character of $W_G(L, \lambda)$,

to χ_μ, constituent of $R_L^G(\lambda)$ (up to sign),

where μ^* is e-quotient of μ.

Similarly, have Isometry I_L^M, M=e-split Levi subgroup containing L,

can choose $M = G_m \times GL(k, q^e)$.

Then: $R_M^G I_L^M = I_L^G \ \mathrm{Ind}_{W_M(L,\lambda)}^{W_G(L,\lambda)}$.

Let $L = G_n \times \mathrm{GL}(k, q^e)$, an e-split Levi subgroup of G_{n+k}. If $\mu \vdash k$, define the Lusztig functor $\mathcal{L}_\mu$ on $[\mathcal{A}]$ where $\mathcal{A} = \oplus_{n \geqslant 0} \mathcal{A}_n$, $\mathcal{A}_n$ the category of unipotent representations of G_n.

$\mathcal{L}_\mu(\chi_\lambda) = R_L^{G_{n+ke}}(\chi_\lambda \times \chi_\mu)$ where $L = G_n \times GL(k, q^e)$, and λ, μ are partitions of n, k respectively.

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Let $\Gamma_n = \mathbf{Z}_e \wr S_n$, complex reflection group.

$\mathrm{Rep}(\Gamma_n)$ = Category of representations of Γ_n over $\mathbb{C}$ and

$\mathrm{Rep}(\Gamma) = \oplus_n \mathrm{Rep}(\Gamma_n)$.

Then $[\mathrm{Rep}(\Gamma)]$ has basis indexed by e-tuples of partitions.

Parabolic subgroup $\Gamma_{n,k}$ of Γ_{n+k} is of the form $\Gamma_n \times \mathcal{S}_k$ where $\mathcal{S}_k$ is a symmetric group.

Have Induction $\mathrm{Rep}(\Gamma_{n,k}) \to \mathrm{Rep}(\Gamma_{n+k})$.

Reference: Shan-Vasserot, p.1010; Uglov 4.1, 4.2

Fock space $\mathcal{F}$ a vector space over $\mathbb{C}$ with standard basis

$\mathcal{B}_1 = \{|\lambda>\}$ indexed by all partitions of $n \geqslant 0$.

There is also a $\mathbb{C}$-basis

$\mathcal{B}_2 = \{(\lambda_e, s_e)\}$ where λ_e runs over e-tuples of partitions, s_e is an e-tuple of integers summing up to 0.

Remark: Regard λ_e as e-quotient, s_e as a label for an e-core of λ. Both can be obtained from the Young diagram of λ.

$\mathcal{A}_n$ = category of unipotent representations of G_n.

If $\mathcal{A} = \oplus_{n \geqslant 0} \mathcal{A}_n$, $[\mathcal{A}]$ (complexified Grothendieck group) is isomorphic to $\mathcal{F}$ as a $\mathbb{C}$-vector space, since $[\mathcal{A}]$ also has a basis indexed by partitions.

$[\mathrm{Rep}(\Gamma)]$ isomorphic to $\mathcal{F}^{(s)}$, subspace of $\mathcal{F}$ with basis (λ_e, s) for fixed s. Both have bases running over e-tuples of partitions.

Heisenberg Lie algebra $\mathfrak{h}$, generators $\langle B_k | k \in \mathbf{Z} - \{0\} \rangle$

with relations $[B_k, B_\ell] = k \frac{1-q^{-2nk}}{1-q^{-2k}} \delta_{k,-\ell}$

Leclerc-Thibon: Commuting operators V_k $(k \geqslant 1)$ in $\mathfrak{h}$ acting on $\mathcal{F}$.

$$V_k(|\lambda >) = \sum_{\mu} (-1)^{-s(\mu/\lambda)} |\mu >$$

where the sum is over all μ such that μ is obtained from λ by adding k e-skew hooks, such that the tail of each skew hook is not upon the head of another skew hook.

s is the leg length of the skew hook.

More generally, we have $V_\rho \in U(\mathfrak{h})$ where ρ is a composition:

If $\rho = \{\rho_1, \rho_2, \ldots\}$ then $V_\rho = V_{\rho 1}.V_{\rho_2} \ldots.$

Then $S_\mu = \sum_\rho k_{\mu\rho} V_\rho$, operator in $U(\mathfrak{h})$, $k_{\mu\rho}$ are inverse Kostka polynomials.

$U(\mathfrak{h})$ acts on $[\mathcal{A}] \leftrightarrow \mathcal{F}$ by S_μ (basis $\mathcal{B}_1$ of partitions, indexing unipotent characters).

Also, $U(\mathfrak{h})$ acts on $\mathcal{F}^{(s)} \leftrightarrow [\mathrm{Rep}(\Gamma)]$ by S_μ, (now on basis $\mathcal{B}_2$ of e-tuples of partitions, indexing $\mathrm{Rep}(\Gamma)$.

Main theorems:

Theorem. S_μ, acting on $\mathcal{F}$, can be identified with Lusztig induction $\mathcal{L}_\mu$ on $[\mathcal{A}]$.

Theorem (Shan-Vasserot) Action of S_μ on $\mathcal{F}^{(s)}$ is identified with ordinary induction in $[\mathrm{Rep}(\Gamma)]$.

Theorem. S_μ, acting on $\mathcal{F}$, can be identified with Lusztig induction $\mathcal{L}_\mu$ on $[\mathcal{A}]$.

Two applications:

(1) Interpretation of BMM bijection

(2) Connection between some Brauer characters and Lusztig induction

(1) Interpretation of BMM Bijection:

Consider the map $\lambda \to (\lambda^*, s)$ where λ^* is the e-quotient of λ and s labels the e-core of λ, between the basis $\mathcal{B}_1$ of all partitions $|\lambda>$ and the basis $\mathcal{B}_2$ of (λ_e, s_e) where λ_e are ℓ-tuples of partitions.

Fix $k, \mu \vdash k$. The action of $S_\mu \in U(\mathfrak{h})$ on $\mathcal{B}_1$, interpreted as on $[\mathcal{A}]$, corresponds to Lusztig induction on the groups G_n. On the other hand, the action on $\mathcal{B}_2$, interpreted as on $[\mathrm{Rep}\Gamma]$, corresponds to ordinary induction on complex reflection groups.

Work done on blocks and decomposition matrices of finite reductive groups: Dipper-James, Geck, Gruber, Hiss, Kessar, Malle .. e.g. modular Harish-Chandra theory.

In Dipper-James theory, have q-Schur algebra $\mathcal{S}_n$.

$K, \mathcal{O}, k$, ℓ-modular system

Dipper-James theory: e is the order of q mod ℓ. Here $q \in k$, characteristic ℓ.

The decomposition matrix of $\mathcal{S}_n$ is square, has entries the multiplicities of irreducibles in Weyl modules.

There is a square part of the decomposition matrix of G_n, rows indexed by unipotent characters, columns by Brauer characters.

These two matrices are the same!

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(2) Back to space $\mathcal{F} \leftrightarrow [\mathcal{A}]$ standard basis χ_λ (unipotent characters). Two canonical bases (Leclerc-Thibon, Uglov), analogous to Lusztig's canonical bases.

$G^+(\lambda) = \sum d_{\lambda\mu}\chi_\mu$

$G^-(\lambda) = \sum e_{\lambda\mu}\chi\mu$

By the work of Varagnolo-Vasserot on decomposition matrix of $\mathcal{S}_n$, for large ℓ we have:

If $\lambda, \mu \vdash n$, $D = (d_{\lambda\mu})$ is the unipotent part of the decomposition matrix of G_n.

If $E = (e_{\lambda\mu})$, E is the inverse transpose of D.

The columns of D express the unipotent characters of G_n in terms of Brauer characters.

Thus, the rows of E express the Brauer characters of G_n in terms of unipotent characters.

Describe $G^{-}(\lambda) = \sum e_{\lambda\mu}\chi_{\mu}$.

Example of the inverse decomposition matrix E for $n = 4$, $e = 2$:

$$\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 & 0 \\ 31|| & -1 & 1 & 0 & 0 & 0 \\ 22|| & 1 & -1 & 1 & 0 & 0 \\ 211|| & -1 & 0 & -1 & 1 & 0 \\ 1111|| & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$G^{-}(211) = -\chi_4 - \chi_{22} + \chi_{211}$,

$G^{-}(22)$, $G^{-}(211)$, $G^{-}(1111)$ are Brauer characters.

Algorithm exists to compute these decomposition numbers in principle.

We wish to describe some of them by Lusztig induction.

Theorem. Let $\lambda \vdash n$, $\lambda = \mu + e\alpha$, $\mu \vdash m$, $\alpha \vdash k$, and let μ' be e-regular. Then the Brauer character represented by $G^{-}(\lambda)$ is equal to the Lusztig generalized character $R_L^{G_n}(G^{-}(\mu) \times \chi_\alpha)$, where n=m+ke, $L = G_m \times GL(k, q^e)$.

Proof. Leclerc-Thibon have proved that $G^{-}(\lambda) = S_\alpha G^{-}(\mu)$, so the proof follows from $S_\alpha = \mathcal{L}_\alpha$.

An example of a decomposition matrix D for $n = 4$, $e = 4$:

$$\begin{pmatrix} 4|| & 1 & 0 & 0 & 0 \\ 31|| & 1 & 1 & 0 & 0 \\ 211|| & 0 & 1 & 1 & 0 \\ 1111|| & 0 & 0 & 1 & 1 \end{pmatrix}$$

An example of the inverse of a decomposition matrix D for $n = 6$,

$$e = 2: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix}$$

Here the rows are indexed as: $6, 51, 42, 41^2, 3^2, 31^3, 2^3, 2^21^2, 21^4, 1^6$

Source: GAP, MAPLE

In the above matrix:

The rows indexed by $1^6, 2^2 1^2, 3^2, 21^4, 41^2$ have interpretations in terms of $R_L^{G_n}$, with L e-split Levi of the form $GL(3, q^2)$, $GL(2, q) \times GL(2, q^2)$, $GL(4, q) \times GL(1, q^2)$, as Brauer characters.

Row indexed by 3^2: $L = GL(3, q^2) : R_L^G(\chi_3) = \chi_{3^2} - \chi_{42} + \chi_{51} - \chi_6$

Row indexed by $2^2 1^2$ is $R_L^G(\chi_{21})$ and

Row indexed by 1^6 is $R_L^G(\chi_{1^3})$.

Michel Broué's philosophy

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BRAUER=LUSZTIG

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